A Tree-Loop Duality Relation at Two Loops and Beyond

Isabella Bierenbaum

In collaboration with: S. Catani, P. Draggiotis, G. Rodrigo

References:
Catani, Gleisberg, Krauss, Rodrigo, Winter JHEP 0809 (2008) 065
Bierenbaum, Catani, Draggiotis, Rodrigo arXiv:1007.0194 [hep-ph]

MOTIVATION:

IR-singularities between virtual and real contributions cancel after integration
$\rightarrow$ the **duality relation** provides an alternative method to existing ones that make use of this fact. It follows the idea:

Try to express the loop integrals as tree-level integrals which are of the same nature as real-radiation integrals

Since then all integrations are of the same tree-level phase-space type, one hopes for a *more efficient implementation* into a Monte Carlo program

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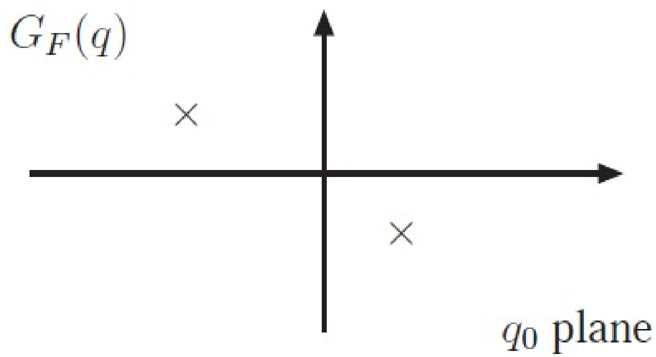
This is currently under investigation for one-loop integrals

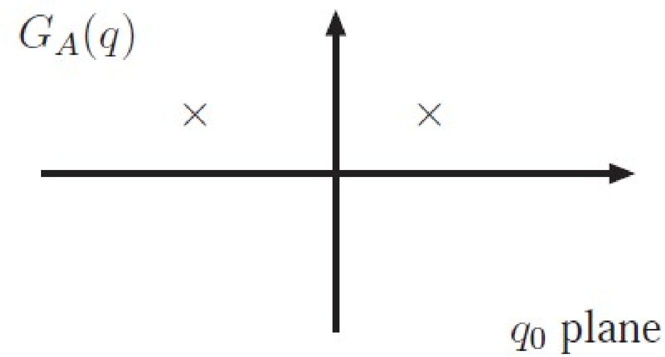
The question for this talk is: If the answer to the above is: „yes, the method is efficient“, is there a way to extend it to higher loop orders?

A one-loop formula in this line of thinking: The ***Feynman tree theorem***

The Feynman Tree Theorem

$$G_F(q) \equiv \frac{1}{q^2 + i0} \quad , \qquad G_A(q) \equiv \frac{1}{q^2 - i0\, q_0}$$



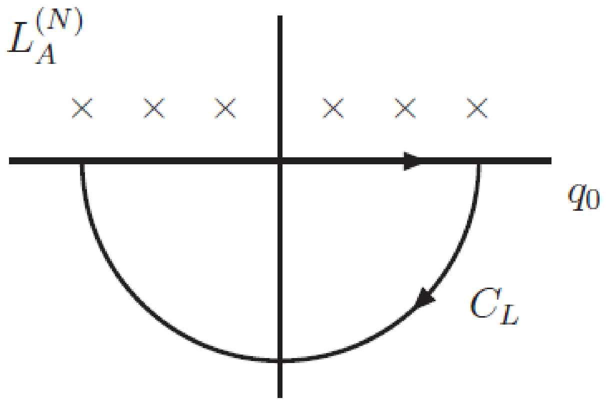


Use: $$\frac{1}{x \pm i0} = \mathrm{PV}\left(\frac{1}{x}\right) \mp i\pi\,\delta(x)$$

$$G_A(q) \equiv G_F(q) + \widetilde{\delta}\,(q) \quad , \qquad G_R(q) \equiv G_F(q) + \widetilde{\delta}\,(-q)$$

$$\widetilde{\delta}\,(q) \equiv 2\pi\, i\,\theta(q_0)\;\delta(q^2) = 2\pi\, i\;\delta_+(q^2)$$

Feynman Tree Theorem



$$G_A(q) \equiv G_F(q) + \widetilde{\delta}(q)$$

$$\begin{aligned} 0 &= L_A^{(N)}(p_1, p_2, \ldots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i) = \int_q \prod_{i=1}^{N} \left[G_F(q_i) + \widetilde{\delta}(q_i) \right] \\ &= L^{(N)}(p_1, p_2, \ldots, p_N) + L^{(N)}_{1-\mathrm{cut}}(p_1, p_2, \ldots, p_N) + \cdots + L^{(N)}_{\mathrm{N-cut}}(p_1, p_2, \ldots, p_N) \end{aligned}$$

$$L^{(N)}_{\mathrm{m-cut}}(p_1, p_2, \ldots, p_N) = \int_q \left\{ \widetilde{\delta}(q_1) \ldots \widetilde{\delta}(q_m) \; G(q_{m+1}) \ldots G(q_N) + \text{uneq. perms.} \right\}$$

Feynman Tree Theorem:

$$L^{(N)}(p_1, p_2, \ldots, p_N) = - \left[L^{(N)}_{1-\mathrm{cut}}(p_1, p_2, \ldots, p_N) + \cdots + L^{(N)}_{\mathrm{N-cut}}(p_1, p_2, \ldots, p_N) \right]$$

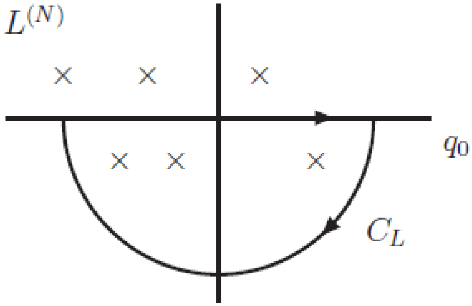
The goal is:

Search for a loop-tree duality relation where
the amount of cuts = number of loops (unlike FTT)

For **one loop** that means:

One single cut is enough !

Towards a *Duality Theorem*: One Loop



Take the integral directly over the residues

$$L^{(N)}(p_1,p_2,\ldots,p_N) \;=\; -2\pi i \int_{\mathbf{q}} \sum \left[\mathrm{Res}_{\{i\mathrm{-th\ pole}\}}\, G_F(q_i)\right] \left[\prod_{j=1,\, j\neq i}^{N} G_F(q_j)\right]_{\{i\mathrm{-th\ pole}\}}$$

$$\left[\mathrm{Res}_{\{i\mathrm{-th\ pole}\}}\, G_F(q_i)\right] \;=\; \left[\mathrm{Res}_{\{i\mathrm{-th\ pole}\}}\, \frac{1}{q_i^2+i0}\right] = \int dq_0\; \delta_+(q_i^2)$$

$$\left[\prod_{j\neq i} G_F(q_j)\right]_{\{i\mathrm{-th\ pole}\}} \;=\; \left[\prod_{j\neq i} \frac{1}{q_j^2+i0}\right]_{\{i\mathrm{-th\ pole}\}} = \prod_{j\neq i} \frac{1}{q_j^2 - i0\,\eta(q_j-q_i)}$$

η is a *future-like* vector, $\eta_\mu = (\eta_0, \boldsymbol{\eta})$, $\eta_0 \geq 0,\ \eta^2 = \eta_\mu \eta^\mu \geq 0$

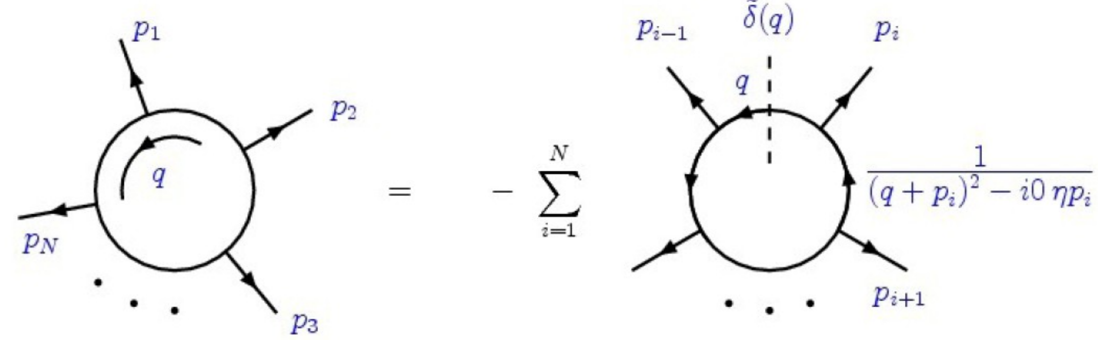
$$G_D(q_i,q_j) := \frac{1}{q_j^2 - i0\,\eta(q_j-q_i)}$$

Dual Propagator

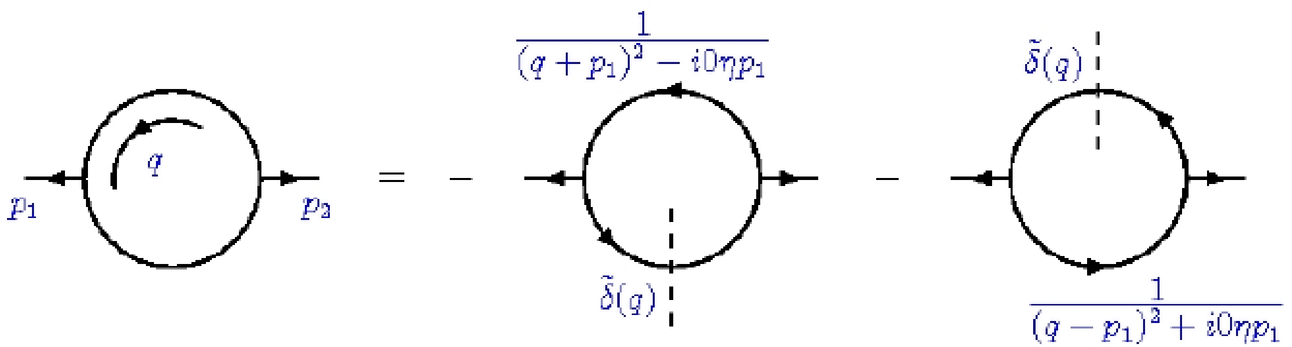
Duality Theorem at one loop

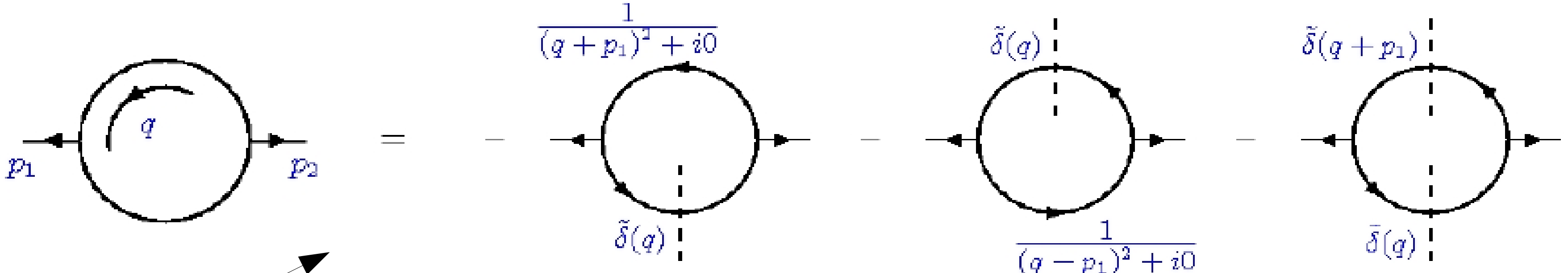
$$L^{(N)}(p_1,p_2,\ldots,p_N) \;=\; -\sum \int_q \tilde{\delta}(q_i) \prod_{j=1,\, j\neq i}^{N} G_D(q_i,q_j)$$

Duality theorem:



Duality theorem:





Feynman tree theorem

At one-loop order:

$$G_D(q_i, q_j) := \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$

- Only single cuts
- The i0-prescription of the dual propagator depends on external momenta only
  $\rightarrow$ no branch cuts

Can we obtain a similar formula at higher loops?

- # cuts = # loops
- integration-momentum independent i0-prescription

At higher loop orders there is more than one integration momentum which we will use to group the diagrams into parts

We start by constructing formulae similar to the once used so far,
but for whole sets of inner momenta

In analogy to single propagators, define for *any set of (internal) momenta α_k*:

$$G_{F(A,R)}(\alpha_k) = \prod_{i\in\alpha_k} G_{F(A,R)}(q_i)$$

$$G_D(\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(q_i;q_j)$$

$$G_D(\alpha_k) = \tilde{\delta}(q_i) \quad \text{for} \quad \alpha_k = \{i\}$$

$$G_D(q_i,q_j) := \frac{1}{q_j^2 - i0\,\eta(q_j-q_i)}$$

$G_D(\alpha_k)$ =

$$\left[\tilde{\delta}(q_1)\,G_D(q_1;q_2)G_D(q_1;q_3) + \tilde{\delta}(q_2)\,G_D(q_2;q_1)G_D(q_2;q_3) + \tilde{\delta}(q_3)\,G_D(q_3;q_1)G_D(q_3;q_2)\right]$$

NOTE:

- If the momenta depend on different integration momenta: integration-momentum dependence in i0-prescription
- If the momenta *depend on the same integration momentum*: i0-prescription depends on external momenta only

Hence, we will naturally try to group higher order diagrams into parts depending on the same integration momenta

$$G_F(-\alpha_k) = G_F(\alpha_k)$$
$$G_A(-\alpha_k) = G_R(\alpha_k)$$

$$G_D(-\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(-q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(-q_i;-q_j)$$

Change direction of momentum flow $q_i \longrightarrow -q_i$ for all momenta $q_i \in \alpha_k$

This will become necessary, starting from two loops

Relation between dual propagator and Feynman propagator:

$$\tilde{\delta}\left(q_i\right)\ G_D(q_i;q_j) = \tilde{\delta}\left(q_i\right)\ \left[G_F(q_j) + \tilde{\theta}\left(q_j - q_i\right)\ \tilde{\delta}\left(q_j\right)\right] \quad \text{with} \qquad \tilde{\theta}\left(q\right) = \theta(\eta q)$$

For ANY set of (internal) momenta, one finds:

Main Equation I

$$\boxed{G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k)}$$

↑

Non-trivial relation relying on cancellation of theta-functions

„Multiplication formula": How to express G_D in terms of subsets $\beta_N \equiv \alpha_1 \cup \ldots \cup \alpha_N$

$$G_D(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N) = G_A(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N) - G_F(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N)$$

$$\begin{aligned} G_D(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N) &= \prod_{i=1}^{N} G_A(\alpha_i) - \prod_{i=1}^{N} G_F(\alpha_i) \\ &= \prod_{i=1}^{N} [G_F(\alpha_i) + G_D(\alpha_i)] - \prod_{i=1}^{N} G_F(\alpha_i) \\ &= \sum_{\beta_N^{(1)} \cup \beta_N^{(2)} = \beta_N} \prod_{i_1 \in \beta_N^{(1)}} G_D(\alpha_{i_1}) \prod_{i_2 \in \beta_N^{(2)}} G_F(\alpha_{i_2}) \end{aligned}$$

Main Equation II

Partition of β_N into exactly two sets $\beta_N^{(1)}$ and $\beta_N^{(2)}$, with elements α_i, including the case $\beta_N^{(1)} = \beta_N$ and $\beta_N^{(2)} = \emptyset$ (There is no term with only G_F)

For example: $G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1)\, G_D(\alpha_2) + G_D(\alpha_1)\, G_F(\alpha_2) + G_F(\alpha_1)\, G_D(\alpha_2)$

$$0 = \int_{\ell_1} G_A(\alpha_1) = \int_{\ell_1} \left[G_F(\alpha_1) + G_D(\alpha_1) \right]$$

Where α_1 is the set of all inner lines of the one-loop diagram

Solve for the „Feynman"-part

$$L^{(1)}(p_1, p_2, \ldots, p_N) = - \int_{\ell_1} G_D(\alpha_1) \quad = - \sum_{i \in \alpha_1} \tilde{\delta}\left(q_i\right) \prod_{\substack{j \in \alpha_1 \\ j \neq i}} G_D(q_i; q_j)$$

Original one-loop result

$$L^{(1)}(p_1, p_2, \ldots, p_N) = -\int_{\ell_1} G_D(\alpha_1)$$

Using the multiplication formula for the set α_1
where the elements are given by all single propagators q_i

$\alpha_1 = q_1 \cup \ldots \cup q_N$

we reproduce the FTT at one loop

$$L^{(1)}(p_1, p_2, \ldots, p_N) = -\sum_{\alpha_1^{(1)} \cup \alpha_1^{(2)} = \alpha_1} \int_{\ell_1} \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_1^{(2)}} G_F(q_{i_2})$$

How can we use this to find a formula for higher order loops

with the required properties?

Use Equation I:

The following statement is correct for ANY set of internal momenta depending on a common integration momentum:

$$0 = \int_{\ell_1} G_A(\alpha_1) = \int_{\ell_1} [G_F(\alpha_1) + G_D(\alpha_1)]$$

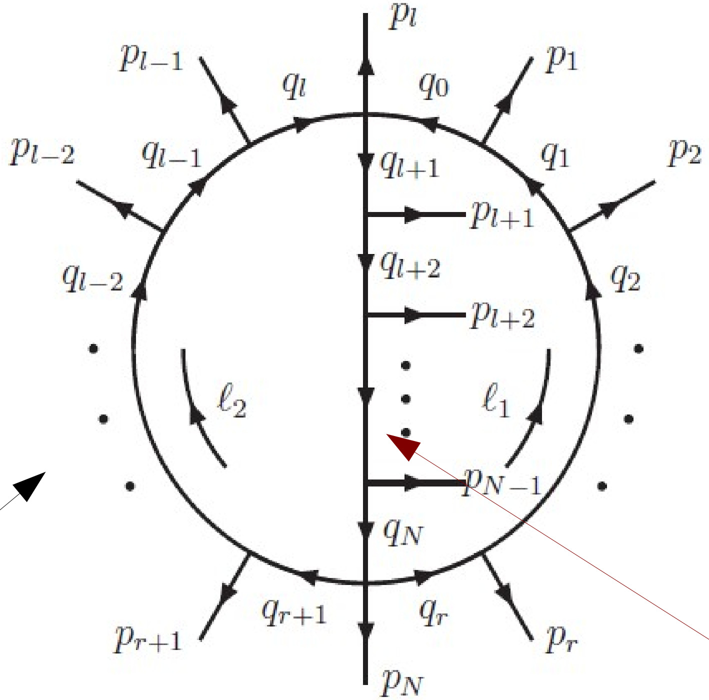
Hence for any set of momenta $\alpha_1 \cup ... \cup \alpha_N$ depending on the same integration momentum :

$$\boxed{\int_{\ell_i} G_F(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) = -\int_{\ell_i} G_D(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N)}$$

Application of the duality theorem!

Two (and higher) loops: ***Find the correct subsets*** and ***use Equation II***

Group lines with the same integration momentum: The „Loop Lines“

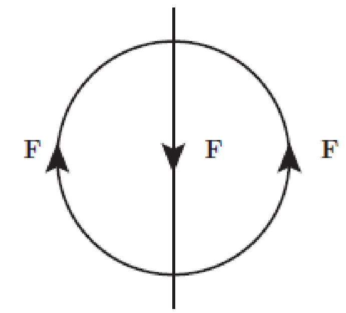


$\alpha_1=\{0,1,\ldots,r\}$

$\alpha_2=\{r+1,\ldots,l\}$

$\alpha_3=\{l+1,\ldots,N\}$

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$



Apply the duality theorem to the first loop

$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{\ell_1} \int_{\ell_2} G_D(\alpha_1 \cup \alpha_3)\, G_F(\alpha_2)$$

Use multiplication formula

$$= -\int_{\ell_1} \int_{\ell_2} \{G_D(\alpha_1)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_F(\alpha_3) + G_F(\alpha_1)\, G_D(\alpha_3)\}\; G_F(\alpha_2)$$

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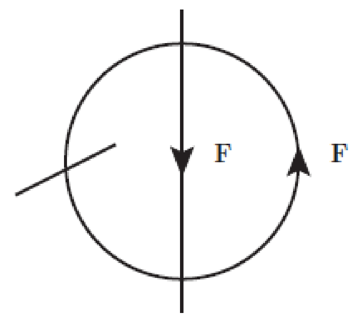
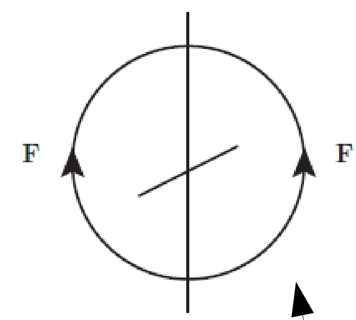
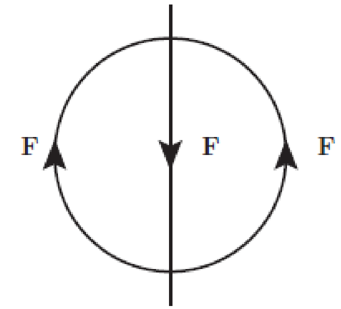




Change direction of momentum-flow for one momentum in this term:

$\alpha_1 \longrightarrow -\alpha_1$



$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{\ell_1}\int_{\ell_2} G_D(\alpha_1 \cup \alpha_3)\, G_F(\alpha_2)$$

$$= \int_{\ell_1}\int_{\ell_2} \{-G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_D(\alpha_2 \cup \alpha_3) + G_D(\alpha_3)\, G_D(-\alpha_1 \cup \alpha_2)\}$$

- Formula with only double-cuts but integration momentum dependent i0-prescription

$$= \int_{\ell_1}\int_{\ell_2} \{G_D(\alpha_1)\, G_D(\alpha_2)\, G_F(\alpha_3) + G_D(-\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G^*(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3)\}$$

$$G^*(\alpha_k) \equiv G_F(\alpha_k) + G_D(\alpha_k) + G_D(-\alpha_k)$$

- Contains triple-cuts but has integration-momentum-free i0-prescription

This can also be expressed as: $G^*(\alpha_k) = G_A(\alpha_k) + G_R(\alpha_k) - G_F(\alpha_k)$

Using again the multiplication formula: **Feynman Tree Theorem at two loops**

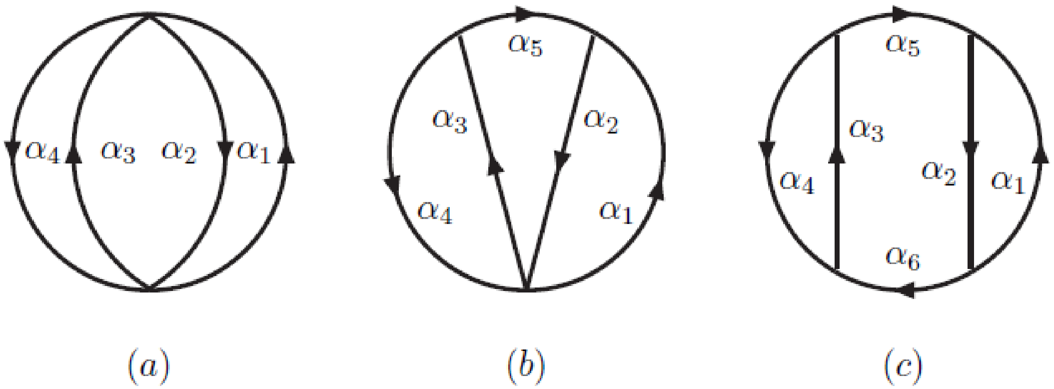
$$L^{(2)}(p_1, \cdots, p_N) = \sum_{\substack{\alpha_k^{(1)} \cup \alpha_k^{(2)} = \alpha_k \\ k \in \{1,2,3\}}} \int_{\ell_1} \int_{\ell_2} \Bigg\{ G_F(\alpha_1) \prod_{i_1 \in \alpha_2^{(1)}} \tilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_3^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_3})$$

$$+ G_F(\alpha_2) \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(-q_{i_1}) \prod_{i_2 \in \alpha_3^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_1^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_3})$$

$$+ G_F(\alpha_3) \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_2^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_1^{(2)} \cup \alpha_2^{(2)}} G_F(q_{i_3})$$

$$+ \left(\prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(q_{i_1}) + \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(-q_{i_1}) \right) \prod_{i_2 \in \alpha_2^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_1 \in \alpha_3^{(1)}} \tilde{\delta}(q_{i_3}) \prod_{i_4 \in \alpha_1^{(2)} \cup \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_4}) \Bigg\},$$

Three loops:

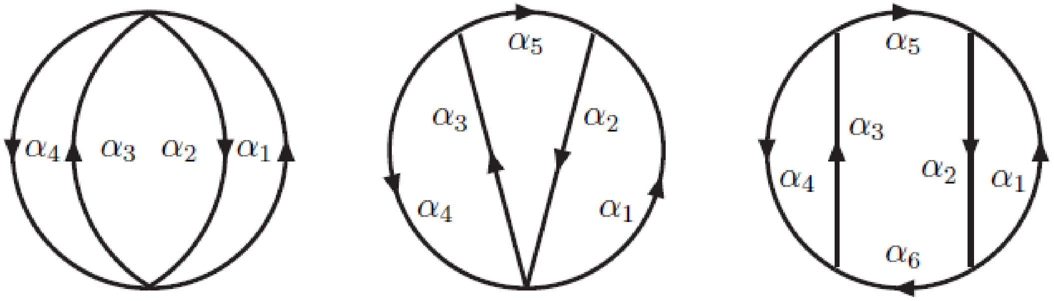


$$L^{(3)}_{(\mathrm{a}),(\mathrm{b}),(\mathrm{c})}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1 \cup \alpha_2)\, G_D(\alpha_3 \cup \alpha_4)\, G_F(\beta)$$

(a): $\beta = \emptyset$ (b): $\beta = \alpha_5$ (c): $\beta = \alpha_5 \cup \alpha_6$

$$G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1)\, G_D(\alpha_2) + G_D(\alpha_1)\, G_F(\alpha_2) + G_F(\alpha_1)\, G_D(\alpha_2)$$

$$\int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1)\, G_F(\alpha_2)\, G_F(\alpha_3)\, G_D(\alpha_4) \;\rightarrow\; - \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1)\, G_D(\alpha_2 \cup \alpha_3)\, G_D(\alpha_4)$$

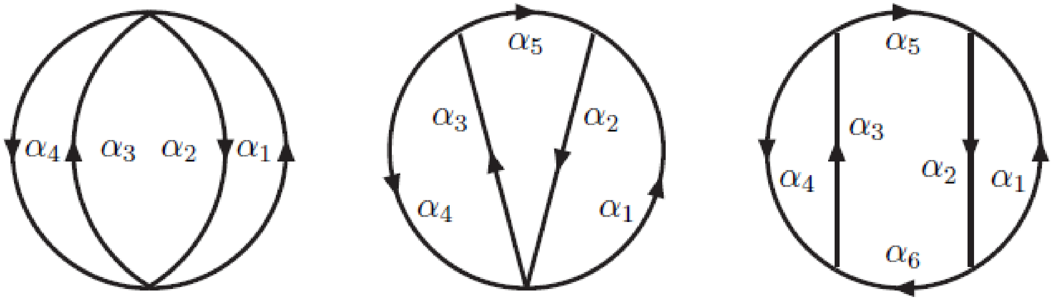


(a) (b) (c)

$$L^{(3)}_{(\mathrm{a}),(\mathrm{b}),(\mathrm{c})}(p_1, p_2, \ldots, p_N) = \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1 \cup \alpha_2)\, G_D(\alpha_3 \cup \alpha_4)\, G_F(\beta)$$

$$= \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Big\{ \Big[G_D(\alpha_2, \alpha_3, \alpha_4)\, G_F(\alpha_1) + G_D(\alpha_1, \alpha_3, \alpha_4)\, G_F(\alpha_2) + G_D(\alpha_1, \alpha_2, \alpha_4)\, G_F(\alpha_3)$$
$$+ G_D(\alpha_1, \alpha_2, \alpha_3)\, G_F(\alpha_4) + G_D(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \Big] G_F(\beta)$$
$$- G_D(\alpha_1, \alpha_3)\, G_D(\alpha_2 \cup -\alpha_4 \cup \beta) - G_D(\alpha_1, \alpha_4)\, G_D(\alpha_2 \cup \alpha_3 \cup \beta)$$
$$- G_D(\alpha_2, \alpha_3)\, G_D(-\alpha_1 \cup -\alpha_4 \cup \beta) - G_D(\alpha_2, \alpha_4)\, G_D(-\alpha_1 \cup \alpha_3 \cup \beta) \Big\}$$

$$G_D(\alpha_1, \ldots, \alpha_N) = \prod_{i=1}^{N} G_D(\alpha_i)$$



$$L^{(3)}_{\text{basket}}(p_1, p_2, \ldots, p_N) = -\int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Big\{ G_D(\alpha_2, \alpha_3, -\alpha_4)\, G_F(\alpha_1) + G_D(\alpha_1, \alpha_3, -\alpha_4)\, G_F(\alpha_2)$$

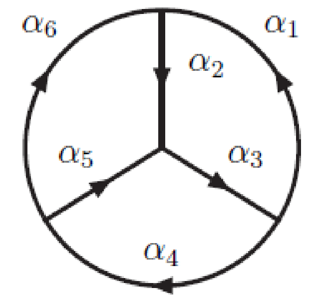
$$+ G_D(-\alpha_1, \alpha_2, \alpha_4)\, G_F(\alpha_3) + G_D(-\alpha_1, \alpha_2, \alpha_3)\, G_F(\alpha_4)$$

$$+ G_D(-\alpha_1, \alpha_2, \alpha_3, \alpha_4) + G_D(\alpha_1, \alpha_2, \alpha_3, -\alpha_4) + G_D(-\alpha_1, \alpha_2, \alpha_3, -\alpha_4) \Big\}$$

Expressed in loop lines (int.-mom.-free i0-prescription):

Cuts, ranging from the *number of loops* → *number of loop lines*

True for any diagram!



$$
\begin{aligned}
&L^{(3)}_{\text{Mercedes}}(p_1, p_2, \ldots, p_N) \\
&\quad = \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} \Big\{ -G_D(\alpha_1, \alpha_2, \alpha_3)\, G_F(\alpha_4, \alpha_5, \alpha_6) + G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)\, G_D(\alpha_1, \alpha_2)\, G_F(\alpha_6) \\
&\quad + G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)\, G_D(\alpha_2, \alpha_3)\, G_F(\alpha_5) + G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)\, G_D(\alpha_1, \alpha_3)\, G_F(\alpha_4) \\
&\quad + G_D(\alpha_1)[G_D(\alpha_3 \cup \alpha_4)\, G_D(\alpha_5)\, G_F(\alpha_2 \cup \alpha_6) - G_D(\alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6)\, G_D(\alpha_5) \\
&\qquad - G_D(\alpha_3 \cup \alpha_4)\, G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)] \\
&\quad + G_D(\alpha_2)[G_D(-\alpha_1 \cup \alpha_6)\, G_D(\alpha_4)\, G_F(\alpha_3 \cup \alpha_5) - G_D(\alpha_1 \cup \alpha_3 \cup \alpha_5 \cup -\alpha_6)\, G_D(\alpha_4) \\
&\qquad - G_D(-\alpha_1 \cup \alpha_6)\, G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)] \\
&\quad + G_D(\alpha_3)[G_D(-\alpha_2 \cup \alpha_5)\, G_D(-\alpha_6)\, G_F(\alpha_1 \cup \alpha_4) - G_D(-\alpha_1 \cup -\alpha_2 \cup \alpha_4 \cup \alpha_5)\, G_D(-\alpha_6) \\
&\qquad\qquad - G_D(-\alpha_2 \cup \alpha_5)\, G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)] \Big\}
\end{aligned}
$$

Conclusions and Outlook:

The efficiency of the method at one loop has to be investigated and is under investigation!

We constructed a loop-tree duality relation which is easily extendable to

higher loop orders, either in the form of

→ n cuts for a n-loop diagram, where the propagators still can involve branch cuts

→ n up to m cuts for a n-loop diagram, with m=#(loop lines), no branch cuts