

Quasinormal modes and isomonodromy

Bruno Carneiro da Cunha — GGI 04/04/22



Outline

- Fuchsian ODEs and flat holomorphic connections;
- Riemann-Hilbert map and monodromy parameters;
- Application to quasi-normal modes;
- Application to uniformizing maps of polycircular domains;

Various works with J. Barragán-Amado, J. P. Cavalcante, T. Anselmo, F. Novaes
S. Nejad, E. Pallante, R. Nelson, D. Crowdy

Preamble

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z), \quad A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}, \quad \Phi(z) = \begin{pmatrix} y_1(z) & y_2(z) \\ w_1(z) & w_2(z) \end{pmatrix}$$



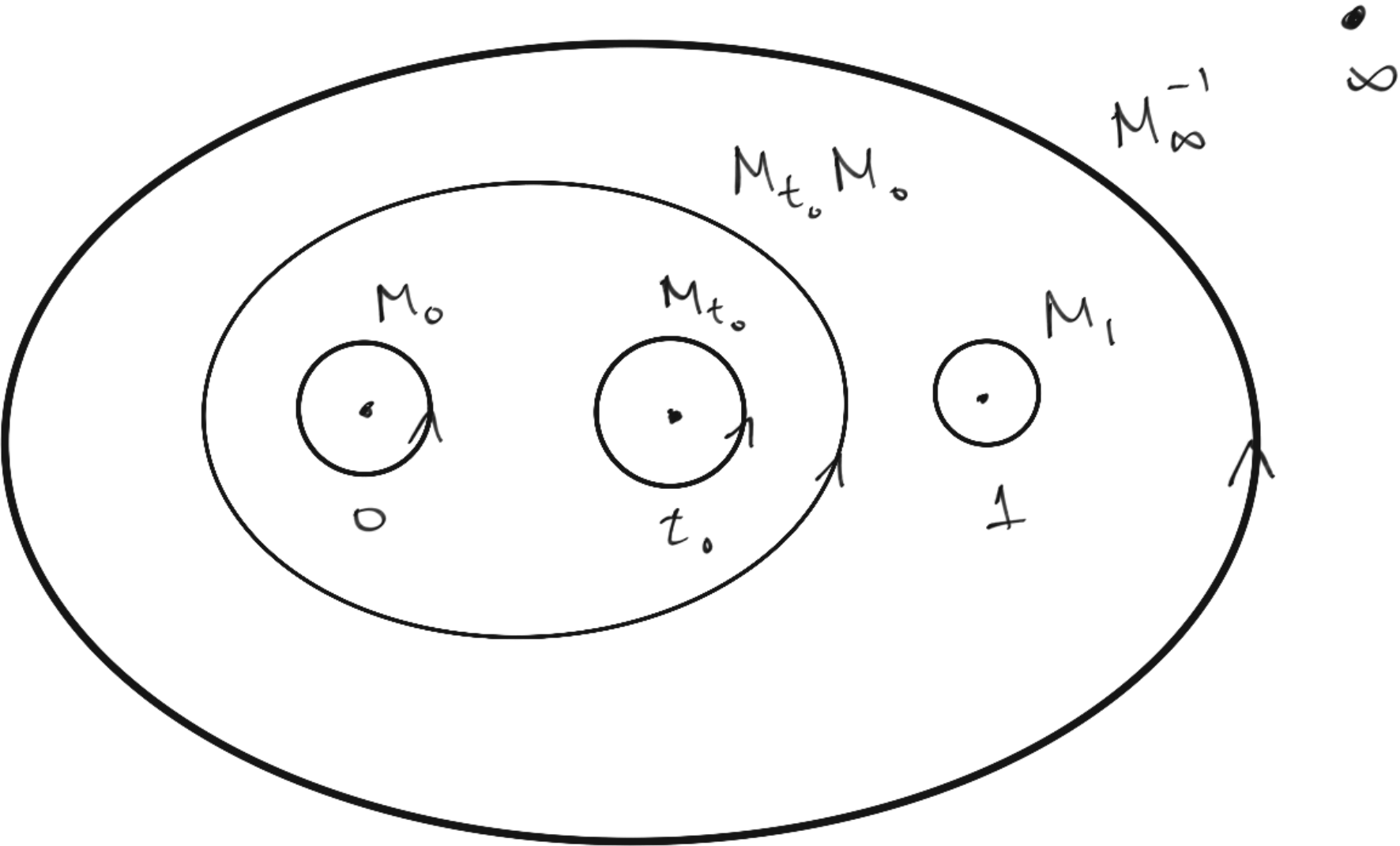
$$\partial_z^2 y - (\operatorname{Tr} A + \partial_z \log a_{12}) \partial_z y + (\det A - \partial_z a_{11} + a_{11} \partial_z \log a_{12}) y = 0.$$

$A(z)$ rational with at most single poles, ODE is Fuchsian. Extra singularity at roots of $a_{12}(z)$.

(Reverse) Riemann-Hilbert problem: how parameters in $A(z)$ affect analytical behavior of solutions?

$$\Phi(z_k + (z - z_k)e^{2i\pi}) = \Phi(z)M_k$$

$$M_k \sim \begin{pmatrix} e^{i\pi\theta_k} & 0 \\ 0 & e^{-i\pi\theta_k} \end{pmatrix}$$



$$M_\infty M_1 M_{t_0} M_0 = \mathbf{1}$$

define Fenchel-Nielsen coordinates

$$\left\{ \begin{array}{l} 2 \cos \pi \sigma = \text{Tr } M_{t_0} M_0 \\ e^{i\eta} \end{array} \right.$$

isomonodromic deformations

“Residual gauge symmetry”: change parameters of $A(z)$ while keeping monodromy data. Schlesinger equations

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_0, A_t], \quad \frac{\partial A_1}{\partial t} = \frac{1}{t-1}[A_1, A_t], \quad \frac{\partial A_t}{\partial t} = -\frac{1}{t}[A_0, A_t] - \frac{1}{t-1}[A_1, A_t],$$

can be translated to the ODE

$$y'' + p(z, t)y' + q(z, t)y = 0,$$

$$p(z, t) = \frac{1 - \hat{\theta}_0}{z} + \frac{1 - \hat{\theta}_1}{z-1} + \frac{1 - \hat{\theta}_t}{z-t} - \frac{1}{z-\lambda}, \quad q(z, t) = \frac{\kappa_1(\kappa_2 + 1)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)},$$

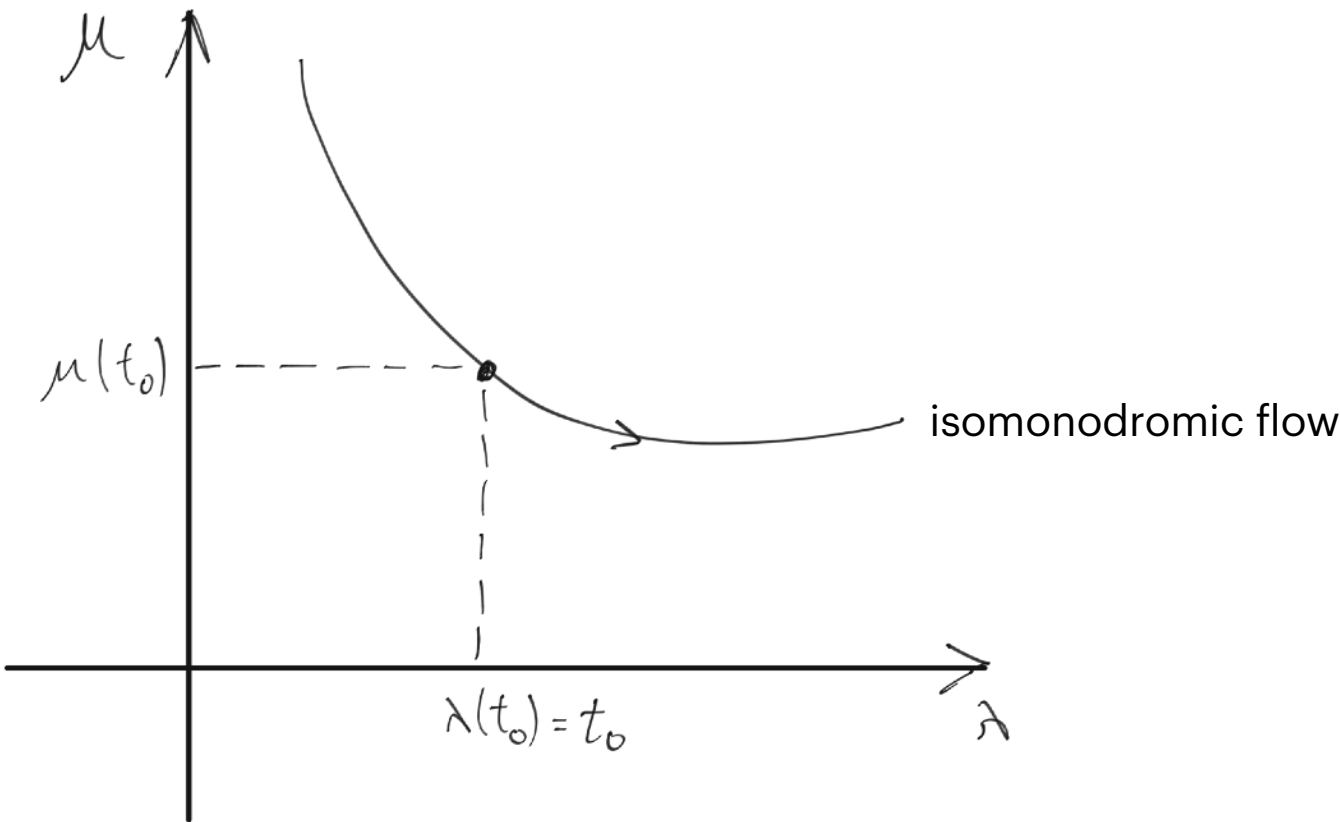
dynamics of apparent singularities maintain monodromy data

The system is Hamiltonian

$$\frac{\partial \lambda}{\partial t} = \frac{\partial K}{\partial \mu}, \qquad \frac{\partial \mu}{\partial t} = - \frac{\partial K}{\partial \lambda},$$

with generating function (Jimbo-Miwa-Ueno tau function)

$$\frac{d}{dt} \log \tau_{JMU}(t) = \frac{1}{t} \text{Tr}(A_0 A_t) + \frac{1}{t-1} \text{Tr}(A_1 A_t),$$



strategy: place initial conditions such that ODE from Schlesinger flow match desired (Heun) ODE

$$\lambda(t_0) = t_0, \qquad \mu(t_0) = \frac{K_0}{\hat{\theta}_{t_0}}$$

Kyiv formula

Full expansion for (isomonodromic, Painlevé VI) tau function is given in terms of monodromy data (GIL2013)

$$\tau(t) = C \sum_m e^{im\tilde{\eta}} \mathcal{B}(\{\theta_k\}, \sigma + 2m; t)$$

$$\mathcal{B}(\{\theta_k\}, \sigma; t) = \mathcal{N}_{\theta_\infty, \sigma}^{\theta_1} \mathcal{N}_{\sigma, \theta_0}^{\theta_t} t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} (1-t)^{\frac{1}{2}\theta_t \theta_1} \sum_{\rho, \nu \in Y} \mathcal{B}(\{\theta_k\}, \sigma) t^{|\rho| + |\nu|},$$

$$\mathcal{B}(\{\theta_k\}, \sigma) = \prod_{(i,j) \in \rho} \frac{((\theta_t + \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 + \sigma + 2(i-j))^2 - \theta_\infty^2)}{4h_\rho^2(i,j)(\rho'_j - i + \nu_i - j + 1 + \sigma)^2} \prod_{(i,j) \in \nu} \frac{((\theta_t - \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 - \sigma + 2(i-j))^2 - \theta_\infty^2)}{4h_\nu^2(i,j)(\nu'_j - i + \rho_i - j + 1 - \sigma)^2}$$

$c=1$ conformal blocks, following AGT. Initial conditions are transcendental equations for ODE parameters

$$\tau(\{\theta_k\}; \sigma, \eta; t_0) = 0,$$

$$K_0 = \frac{\partial}{\partial t} \log \tau(\{\theta_k\}^-; \sigma - 1, \eta; t_0) - \frac{\theta_t - 1}{2t_0} - \frac{\theta_t - 1}{2(t_0 - 1)}$$

Problem generalizes: any Fuchsian ODE Riemann-Hilbert problem can be solved this way

$$A(z; z_k) = \sum_k \frac{A_k}{z - z_k}, \quad \frac{\partial A_k}{\partial z_l} = \frac{[A_k, A_l]}{z_k - z_l}, \quad \frac{\partial A_k}{\partial z_k} = - \sum_{l \neq k} \frac{[A_k, A_l]}{z_k - z_l}$$

(multi-)Hamiltonian flow:

$$\frac{\partial}{\partial z_k} \log \tau(\{\theta_k\}; \{\sigma_k\}, \{\eta_k\}; \{z_k\}) = \sum_{l \neq k} \frac{\text{Tr } A_k A_l}{z_k - z_l}$$

allows transcendental equations that determine accessory parameters from monodromy data

$$\tau_{\text{JMU}}(\hat{\rho}_k^+; \{w_k\}) = 0, \quad \beta_k = - \frac{\partial}{\partial t_k} \log \tau_{\text{JMU}}(\hat{\rho}; \{w_k\}) + \frac{\hat{\theta}_k}{2w_k} + \frac{\hat{\theta}_k}{2(w_k - 1)}.$$

Why monodromy data?

monodromy data determines (partially) connection data

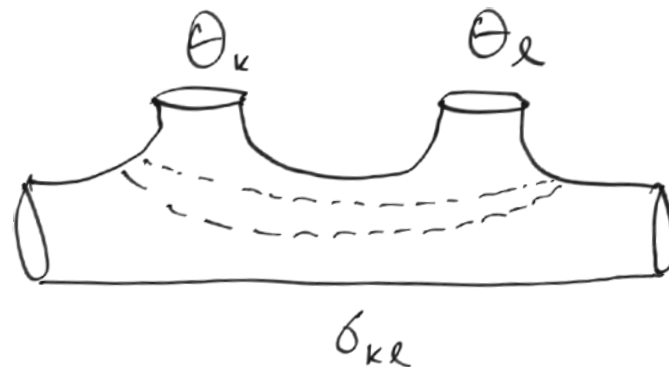
$$\Phi_k(z) = (z - z_k)^{\frac{1}{2}\theta_k\sigma_3}(1 + \mathcal{O}(z - z_k))$$

$$\Phi_k(z) = \Phi_l(z)C_{kl}$$

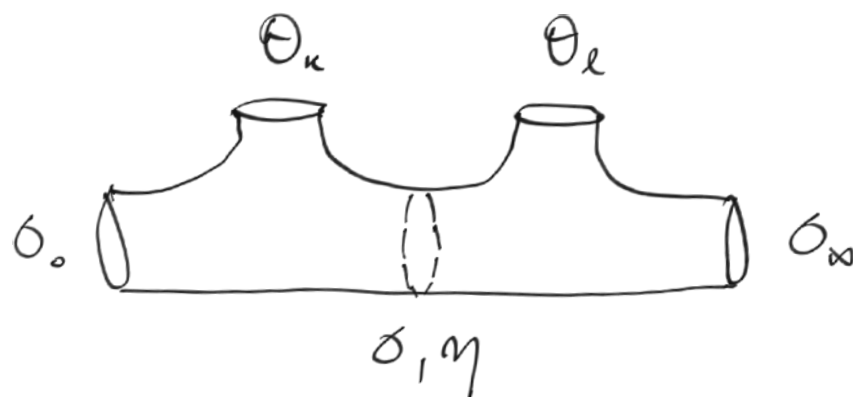
$$M_k = e^{i\pi\theta_k\sigma_3}, \quad M_l = C_{kl}e^{i\pi\theta_l\sigma_3}C_{kl}^{-1},$$

$$\text{Tr } M_k M_l = 2 \cos \pi\sigma_{kl}$$

Example: triangular connection



$$\sigma_{kl} = \theta_k + \theta_l + 2n, \quad n \text{ integer}$$



$$e^{i\pi\eta} = \frac{\sin \frac{\pi}{2}(\theta_l + \sigma_\infty + \sigma) \sin \frac{\pi}{2}(\theta_l - \sigma_\infty + \sigma) \sin \frac{\pi}{2}(\theta_k + \sigma_0 + \sigma) \sin \frac{\pi}{2}(\theta_k - \sigma_0 + \sigma)}{\sin \frac{\pi}{2}(\theta_l + \sigma_\infty - \sigma) \sin \frac{\pi}{2}(\theta_l - \sigma_\infty - \sigma) \sin \frac{\pi}{2}(\theta_k + \sigma_0 - \sigma) \sin \frac{\pi}{2}(\theta_k - \sigma_0 - \sigma)}$$

Conformal blocks realize the Riemann-Hilbert map

$$\{\sigma_i, \eta_j\} = \frac{1}{2\pi} \delta_{ij} \longleftrightarrow \{t_i, K_j\} = \delta_{ij}$$

Semi-classical level 2 null vector condition of Liouville

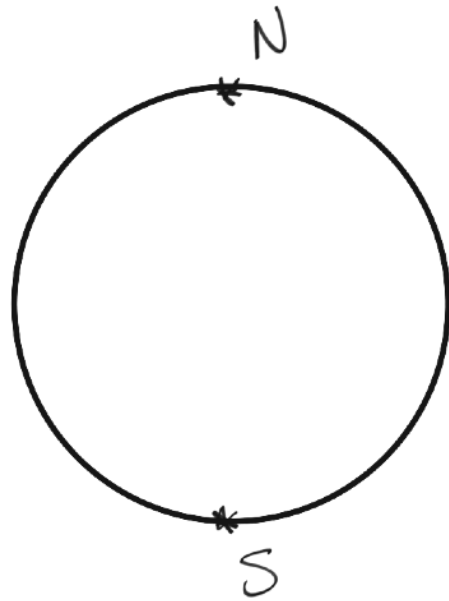
methods of calculation

- Zamolodchikov recursion formula (any b);
- Nekrasov expansions (appearance of $\mathcal{M}(\mathbb{C})$ as moduli of instantons in $\mathcal{N}=2$ SYM) (any b);
- Riemann-Hilbert problem ($b \rightarrow 0$).

Application: black hole QNMs

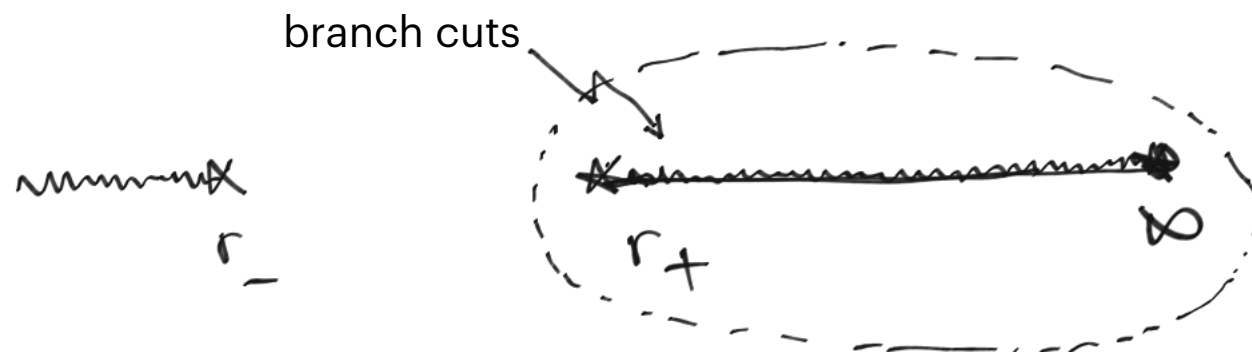
Schematically, wave equation is separable in many black hole backgrounds, for different values of spin

$$\nabla^2 \Phi = \mu^2 \Phi, \quad \Phi = \sum e^{-i\omega t + im\phi} R_{\omega, \ell, m}(r) S_{\omega, \ell, m}(\theta)$$



angular equation determines ℓ by requiring solution to be regular at North and South poles

r



radial QNM determined by requiring solution with no flux at outer horizon and infinity

why one should care?

Analytical & numerical methods exist for a number of black holes (ODE falls into Heun class); BUT...

τ permits a formal solution in terms of monodromy data, confluent limit is well-controlled,
Miwa's theorem guarantees analyticity and isolated zeros;

Relation to conformal blocks is explicit and (perhaps) hints at underlying integrable or physical structure,
resurgence is clear (for PVI) and allows for choice of expansion point;

Numerically stable methods (Fredholm determinant formulation) exist for τ .
Works for higher number of singular points.

Scalar QNMs in 5d Kerr-AdS black hole

with J. Barragán-Amado and E. Pallante

angular part: Heun equation

$$\theta_0 = m_1, \quad \theta_1 = 2 - \Delta, \quad \theta_{u_0} = m_2, \quad \theta_\infty = \omega + a_1 m_1 + a_2 m_2 \quad u = 0, \quad u = 1, \quad u = u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}, \quad u = \infty,$$

$$4u_0(u_0 - 1)Q_0 = -\frac{\omega^2 + a_1^2 \mu^2 - \lambda}{a_2^2 - 1} - u_0 [(m_2 - \Delta + 1)^2 - m_2^2 - 1] - (u_0 - 1) [(1 - m_1 - m_2)^2 - \beta^2 - 1]$$

monodromy condition allows the computation of eigenvalue using Nekrasov expansion

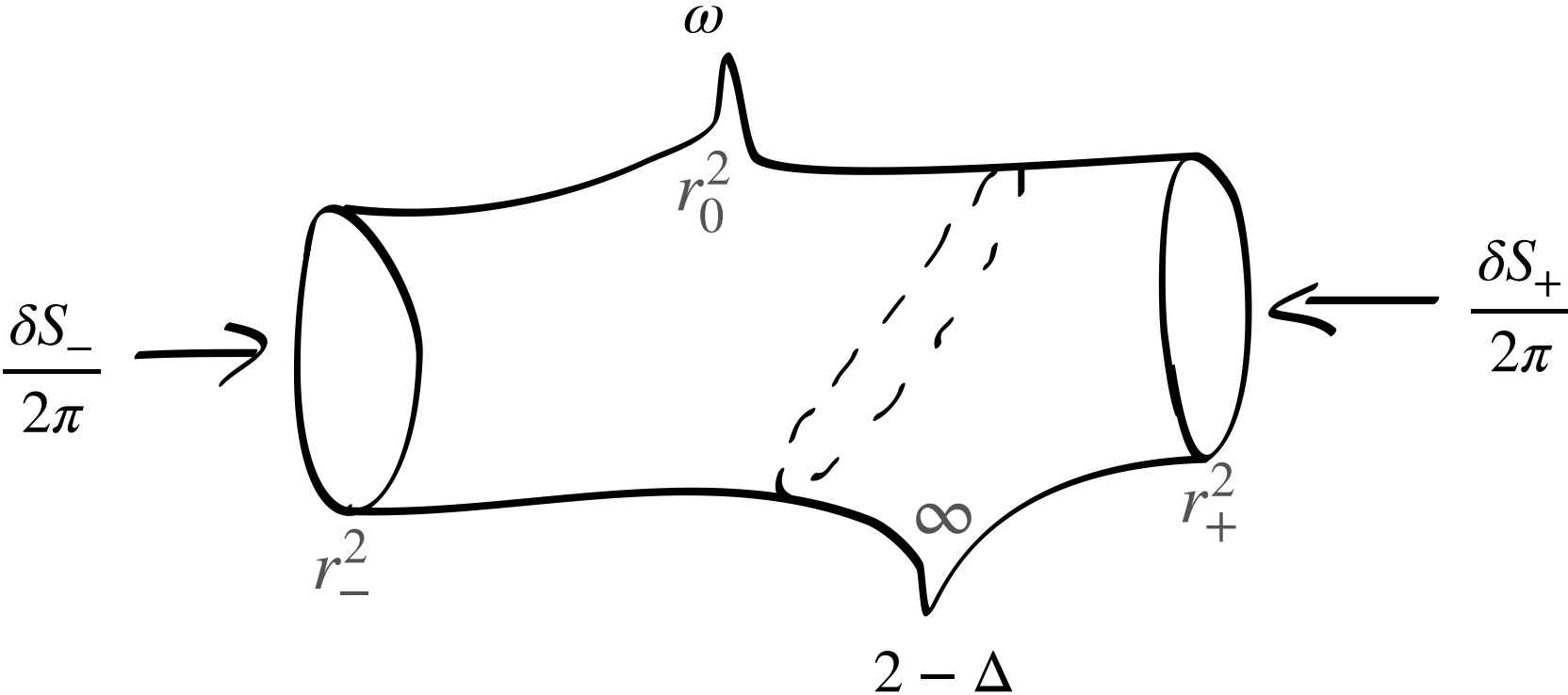
$$\lambda_\ell \simeq \ell(\ell + 2) - 2\omega (a_1 m_1 + a_2 m_2) - (a_1 m_1 + a_2 m_2)^2 + \frac{a_1^2 + a_2^2}{2} (\beta^2 + \mu^2 - \ell(\ell + 2)) \\ + \frac{(a_2^2 - a_1^2) (m_2^2 - m_1^2)}{2\ell(\ell + 2)} (\beta^2 - \mu^2 - (\ell^2 + 2\ell + 4)) + \mathcal{O}((a_2^2 - a_1^2)^2)$$

radial equation: also Heun equation

$$\theta_k = \frac{i}{2\pi} \left(\frac{\omega - m_1 \Omega_{k,a} - m_2 \Omega_{k,b}}{T_k} \right), \quad \theta_\infty = 2 - \Delta, \quad z_0 = (r_+^2 - r_-^2)/(r_+^2 - r_0^2)$$

$$4z_0(z_0 - 1)K_0 = -\frac{\lambda + \mu^2 r_-^2 - \omega^2}{r_+^2 - r_0^2} - (z_0 - 1) [(\theta_- + \theta_+ - 1)^2 - \theta_0^2 - 1] - z_0 [(\theta_+ - \Delta + 1)^2 - \theta_+^2 - 1]$$

Liouville representation (entropy intake)



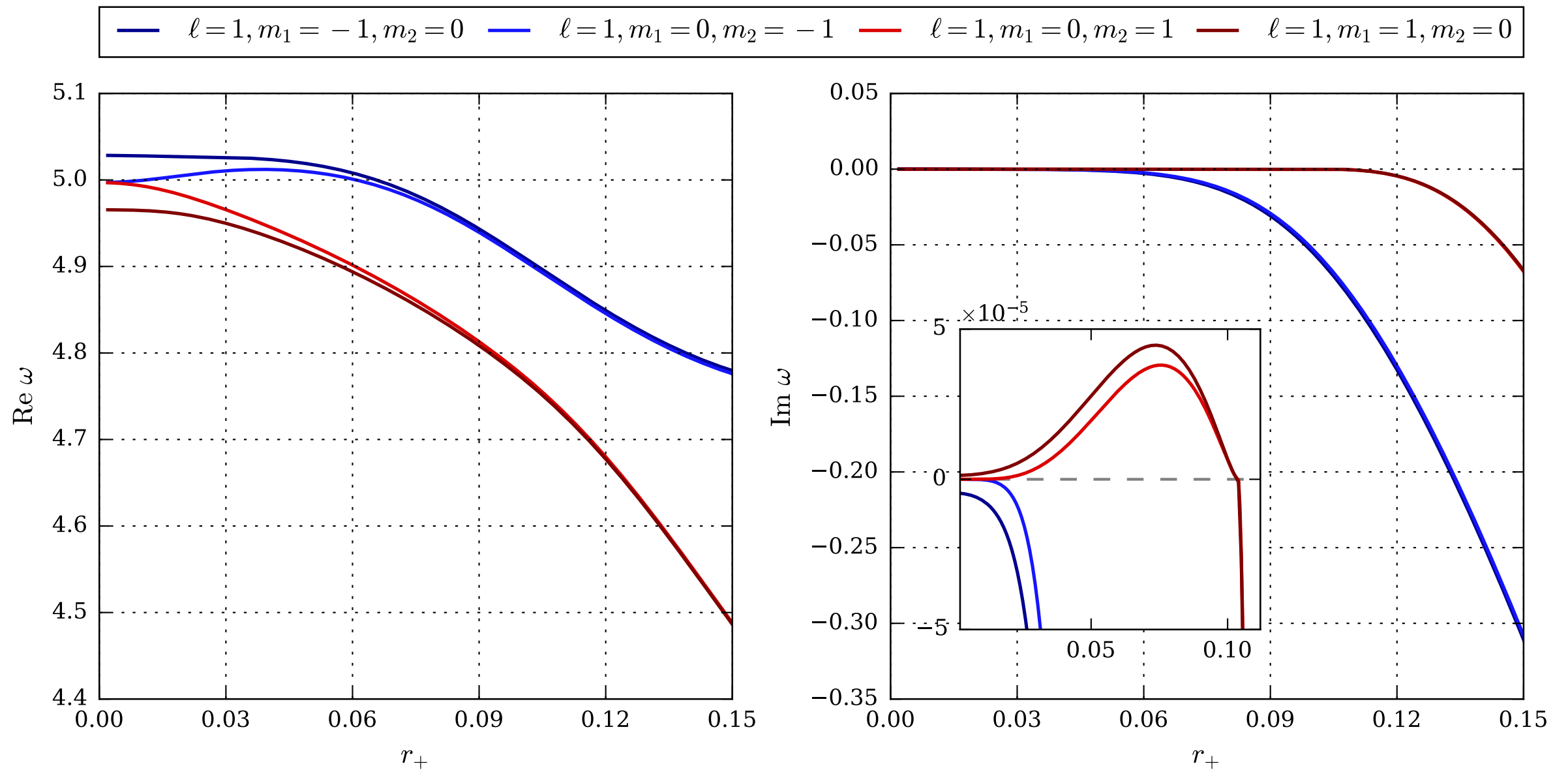
Analytical results for QNMs:

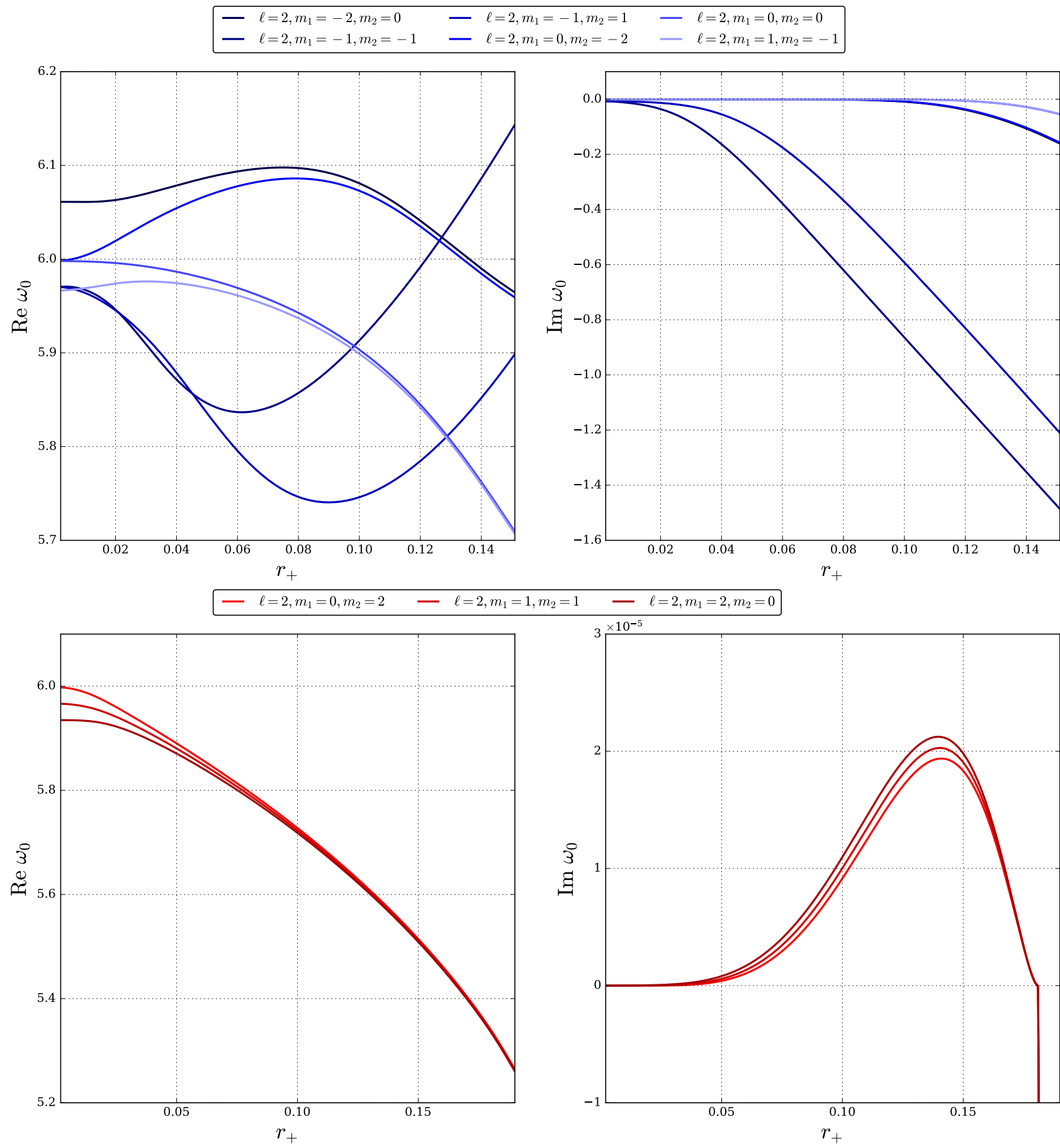
$$\omega_{1,0,0,0} = \Delta - (1 + \alpha_+^2)\Delta(\Delta - 1)r_+^2 - 2i(1 + \alpha_+^2)\Delta(\Delta - 1)r_+^3 + \Delta(\Delta - 1)\epsilon r_+^2 + i(3 + \alpha_+^2)\Delta(\Delta - 1)\epsilon r_+^3 + \mathcal{O}(r_+^4, r_+^4 \log r_+^2, \epsilon r_+^4, \epsilon^2 r_+^2)$$

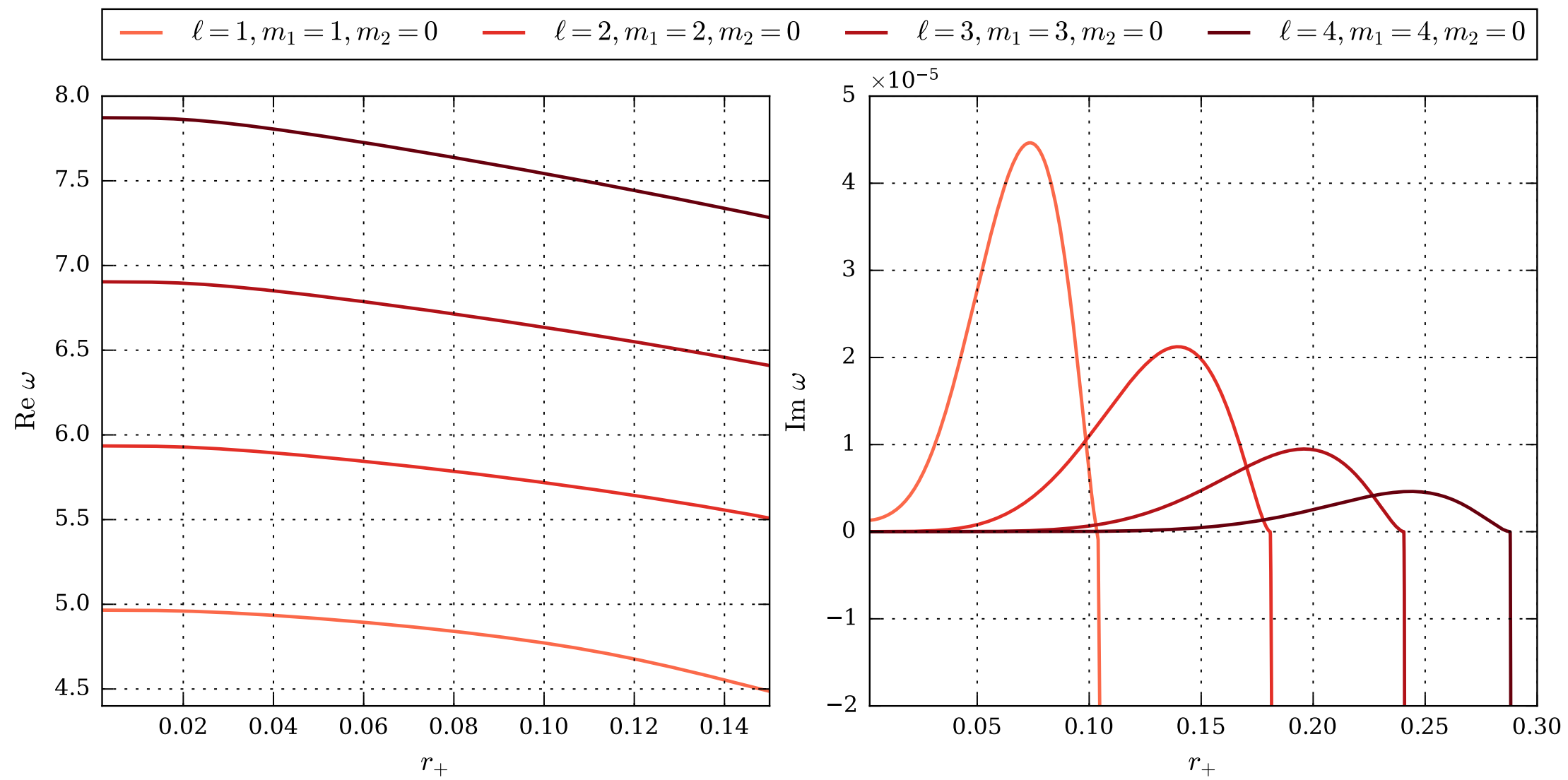
$$\epsilon = \frac{r_+^2 - r_-^2}{2r_+^2}, \qquad \alpha_+^2 = \frac{a_1^2 + a_2^2}{2r_+^2}$$

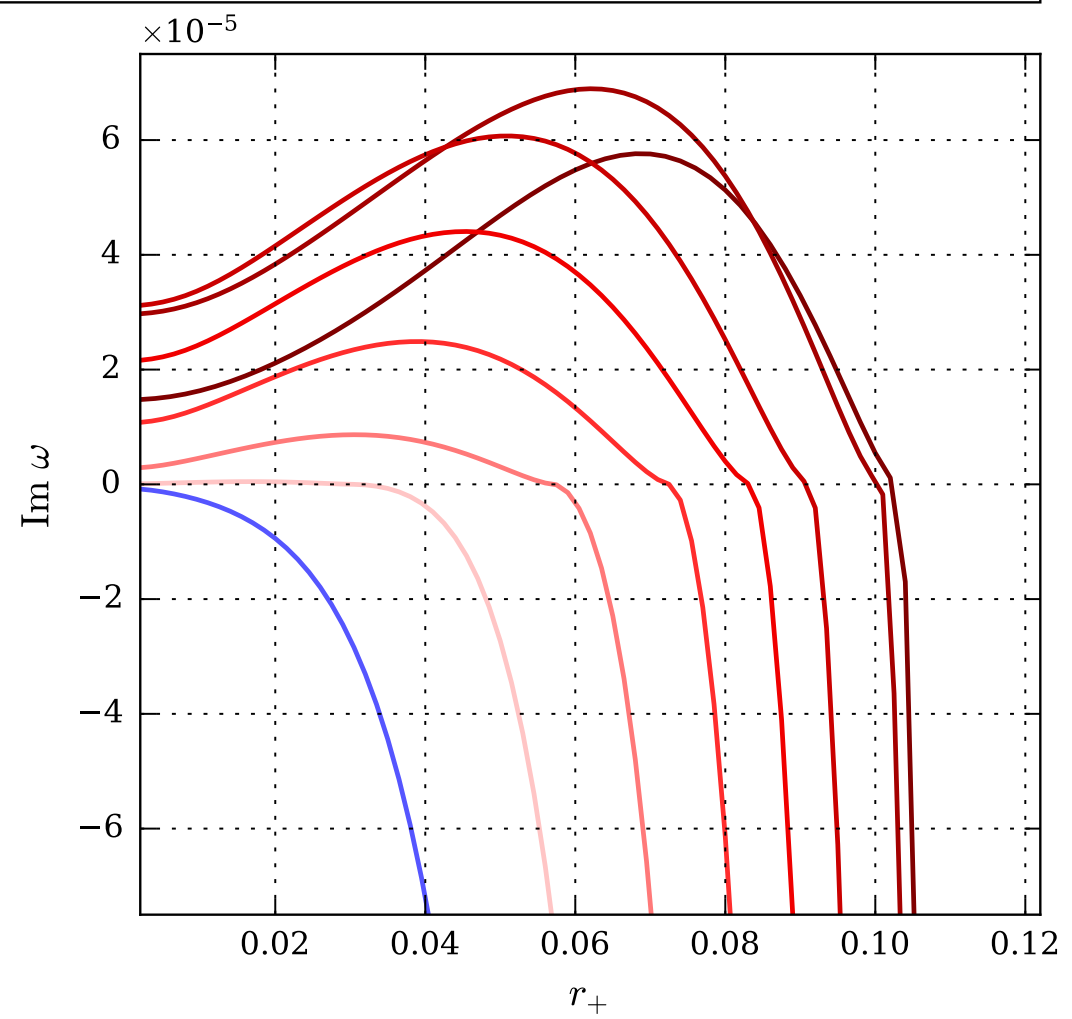
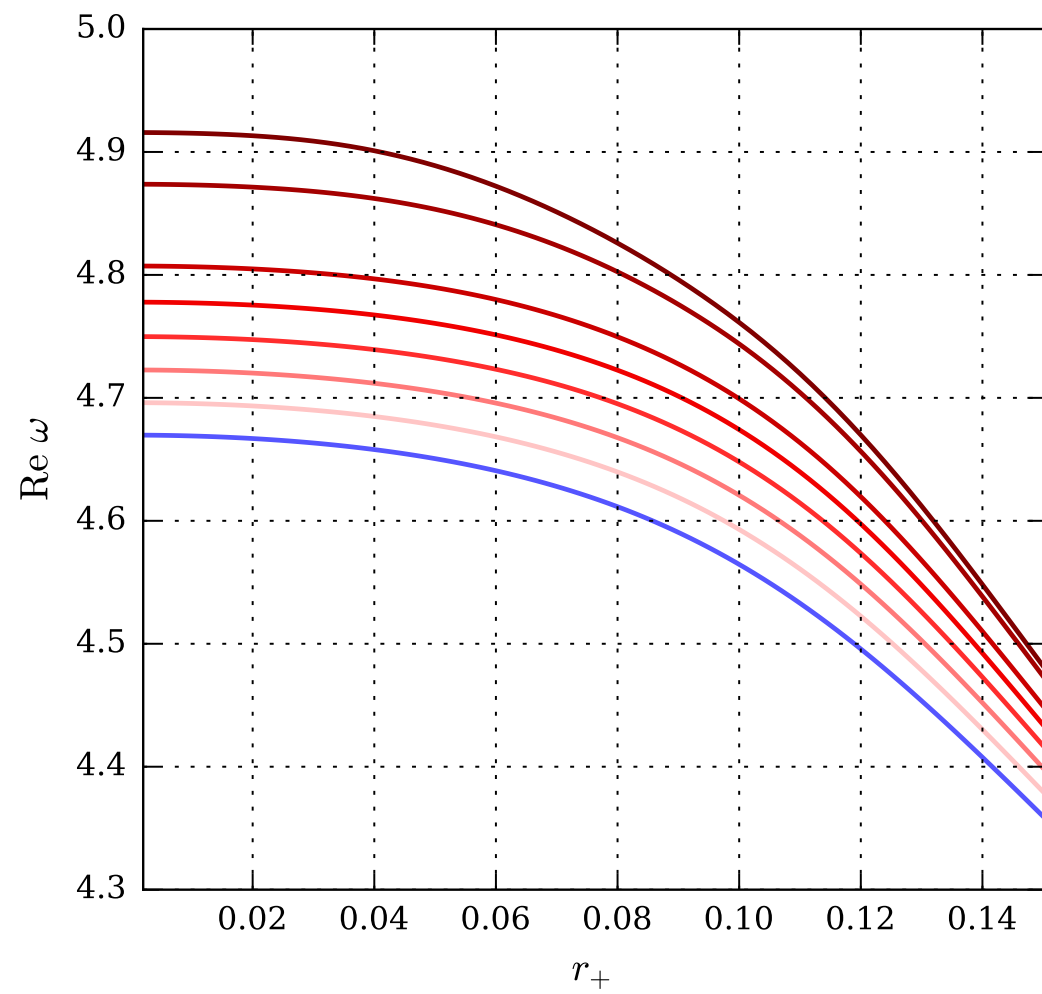
structure of a transseries

numerical results: instabilities





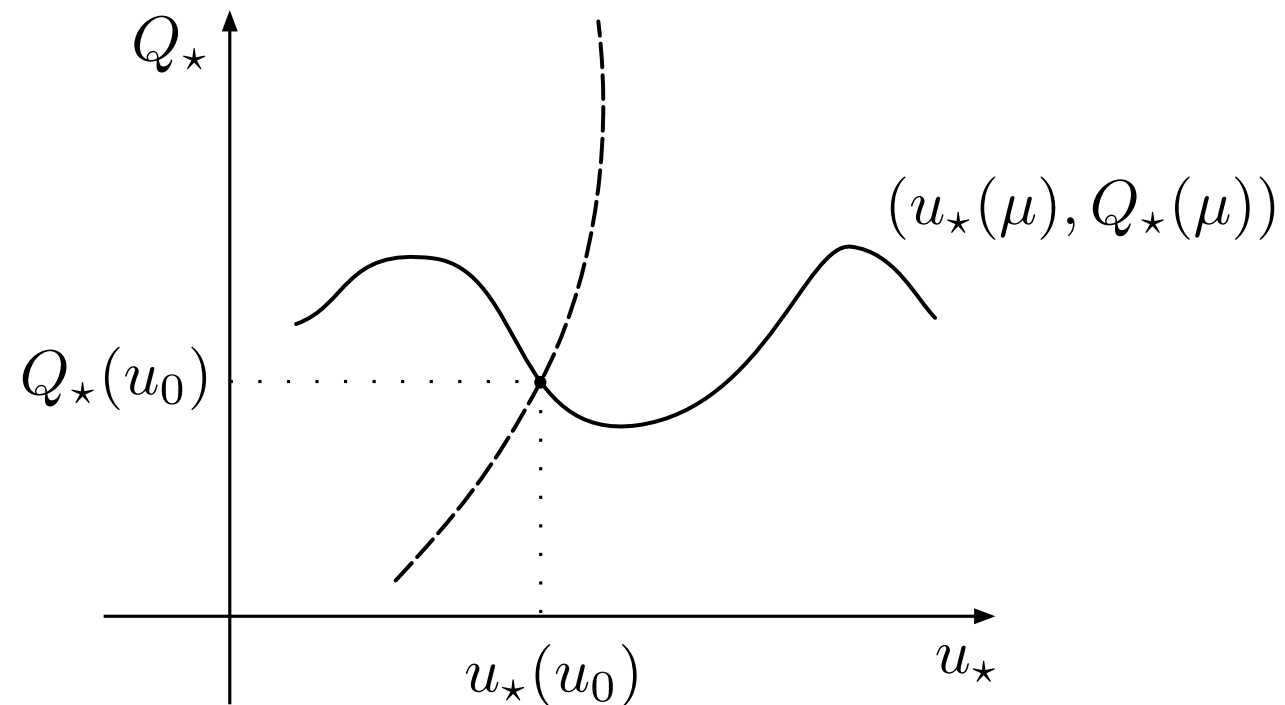




vector perturbations of 5d Kerr-AdS

μ -separability (Lunin, Frolov-Krtouš): resulting ODEs are Heun plus apparent singularity

single monodromy parameters for radial, angular variables now depend on separation constant



initial value problem now sets value for μ , related to polarization of QNM modes

only numerical study, also evidence for superradiance

4d Kerr and Reissner-Nordström black hole

with J. P. Cavalcante

angular equation:

$${}_s\lambda_{\ell,m}(a\omega) = (\ell - s)(\ell + s + 1) - \frac{2ms^2}{\ell(\ell + 1)}a\omega + \left(\frac{2((\ell + 1)^2 - m^2)((\ell + 1)^2 - s^2)^2}{(2\ell + 1)(\ell + 1)^3(2\ell + 3)} - \frac{2(\ell^2 - m^2)(\ell^2 - s^2)^2}{(2\ell - 1)\ell^3(2\ell + 1)} - 1 \right) a^2\omega^2 + \mathcal{O}(a^3\omega^3)$$

radial equation:

$$\theta_{\text{Rad},0} = \theta_- = s - i \frac{\omega - m\Omega_-}{2\pi T_-}, \quad \theta_{\text{Rad},t} = \theta_+ = s + i \frac{\omega - m\Omega_+}{2\pi T_+}, \quad \theta_{\text{Rad},\star} = \theta_* = -2s + 4iM\omega, \quad 2\pi T_{\pm} = \frac{r_+ - r_-}{4Mr_{\pm}}, \quad \Omega_{\pm} = \frac{a}{2Mr_{\pm}}$$

$$t_{\text{Rad}} = z_0 = 2i(r_+ - r_-)\omega, \quad t_{\text{Rad}}c_{\text{Rad},t} = z_0c_0 = {}_s\lambda_{\ell,m} + 2s + 2i(1 - 2s)M\omega - is(r_+ - r_-)\omega + (M^2a^2 - 4Mr_+)\omega^2$$

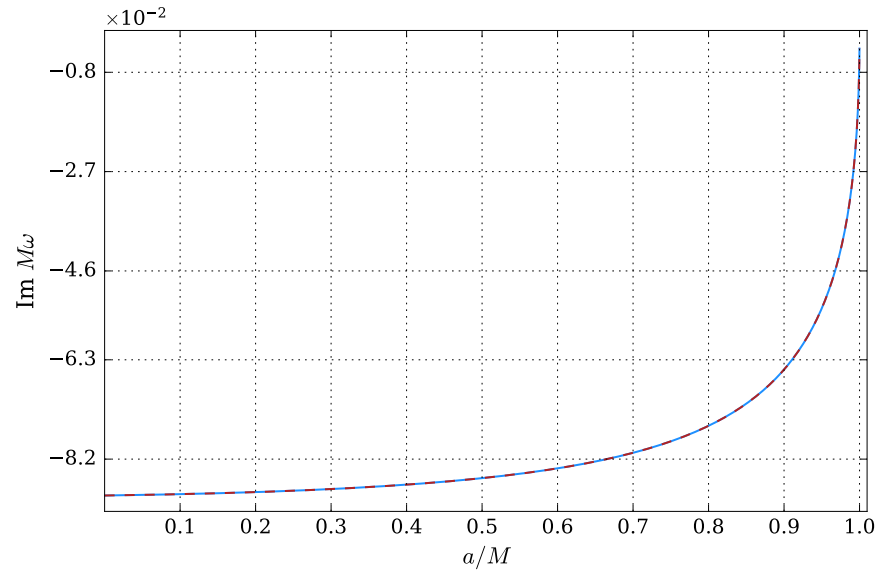
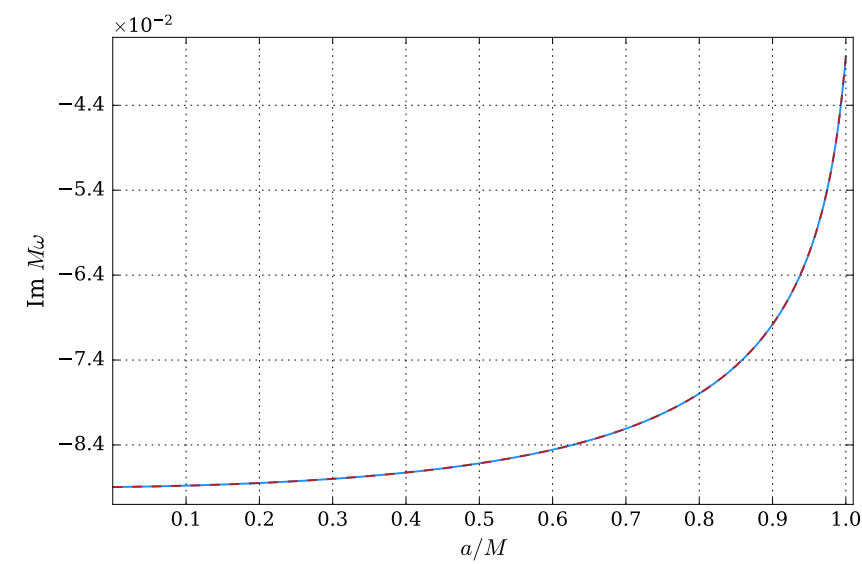
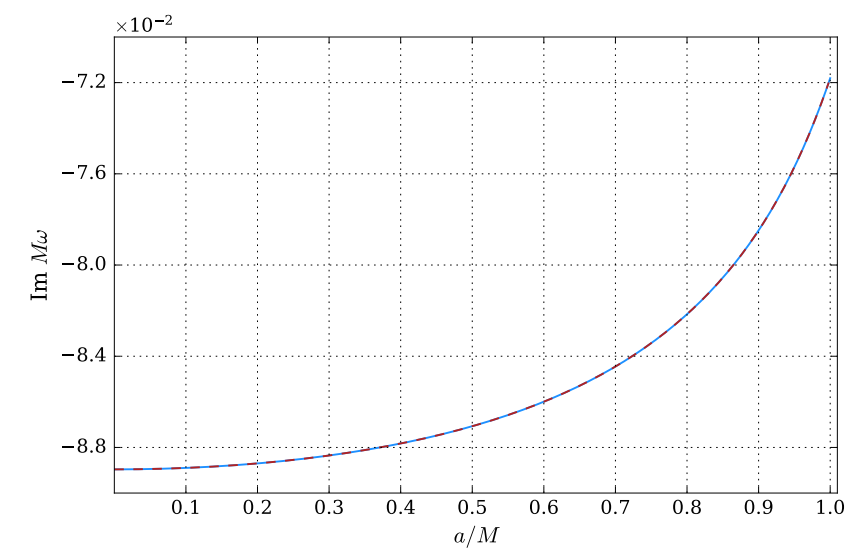
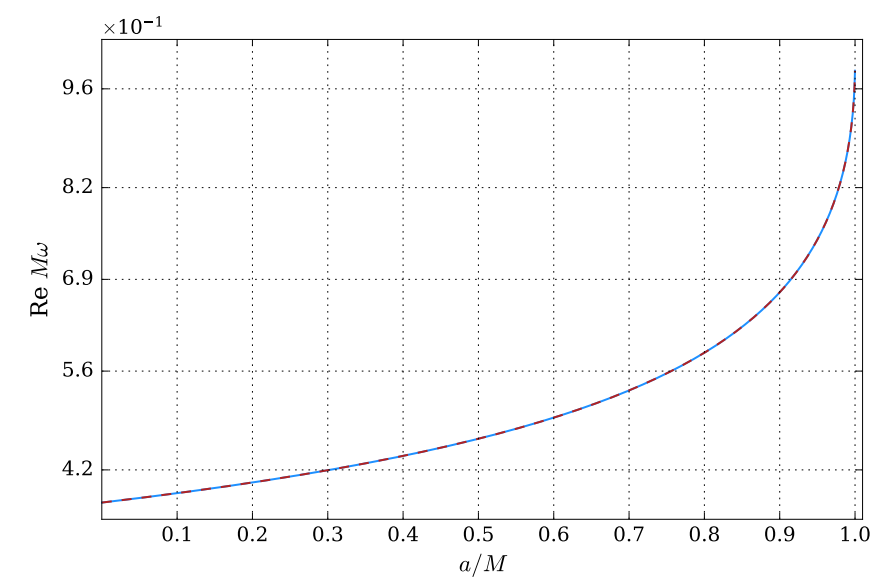
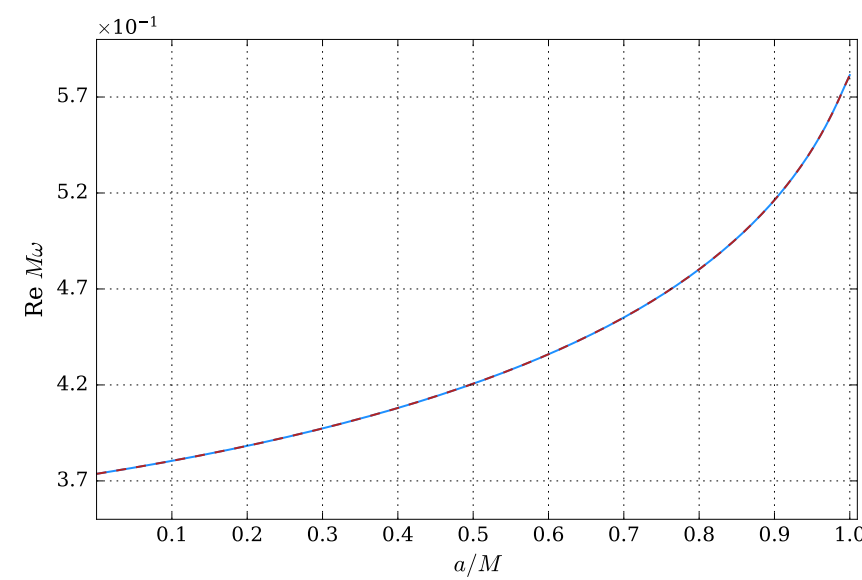
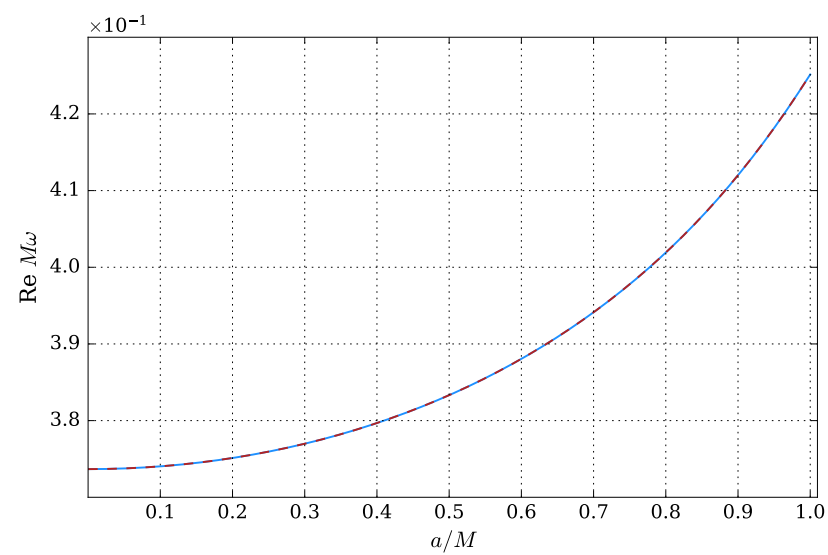
ODE is now confluent Heun, complicated analytical structure near infinity...

$$e^{i\pi\eta_0} = e^{-i\pi\sigma} \frac{\sin \frac{\pi}{2}(\theta_{\star} + \sigma)}{\sin \frac{\pi}{2}(\theta_{\star} - \sigma)} \frac{\sin \frac{\pi}{2}(\theta_t + \theta_0 + \sigma) \sin \frac{\pi}{2}(\theta_t - \theta_0 + \sigma)}{\sin \frac{\pi}{2}(\theta_t + \theta_0 - \sigma) \sin \frac{\pi}{2}(\theta_t - \theta_0 - \sigma)}$$

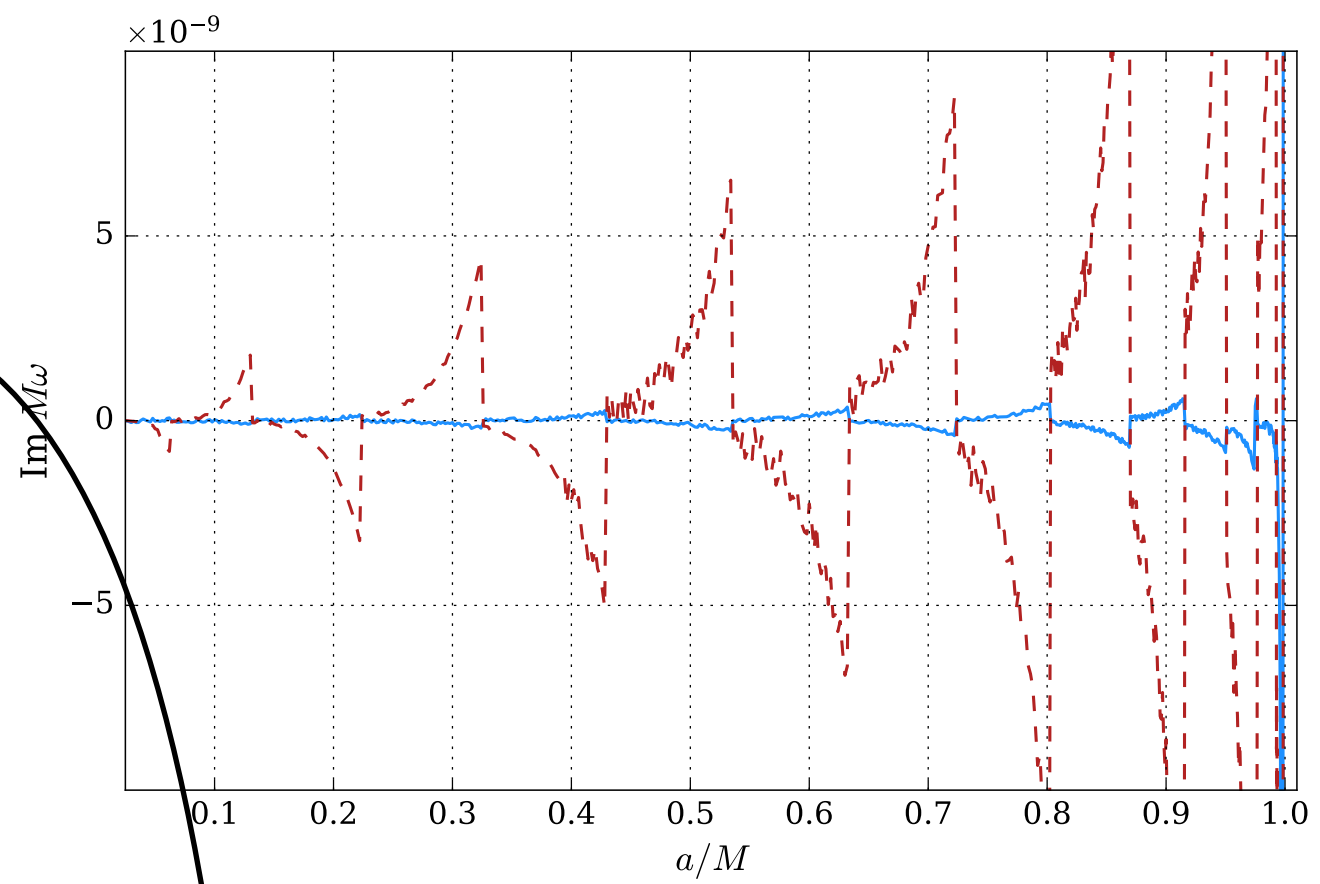
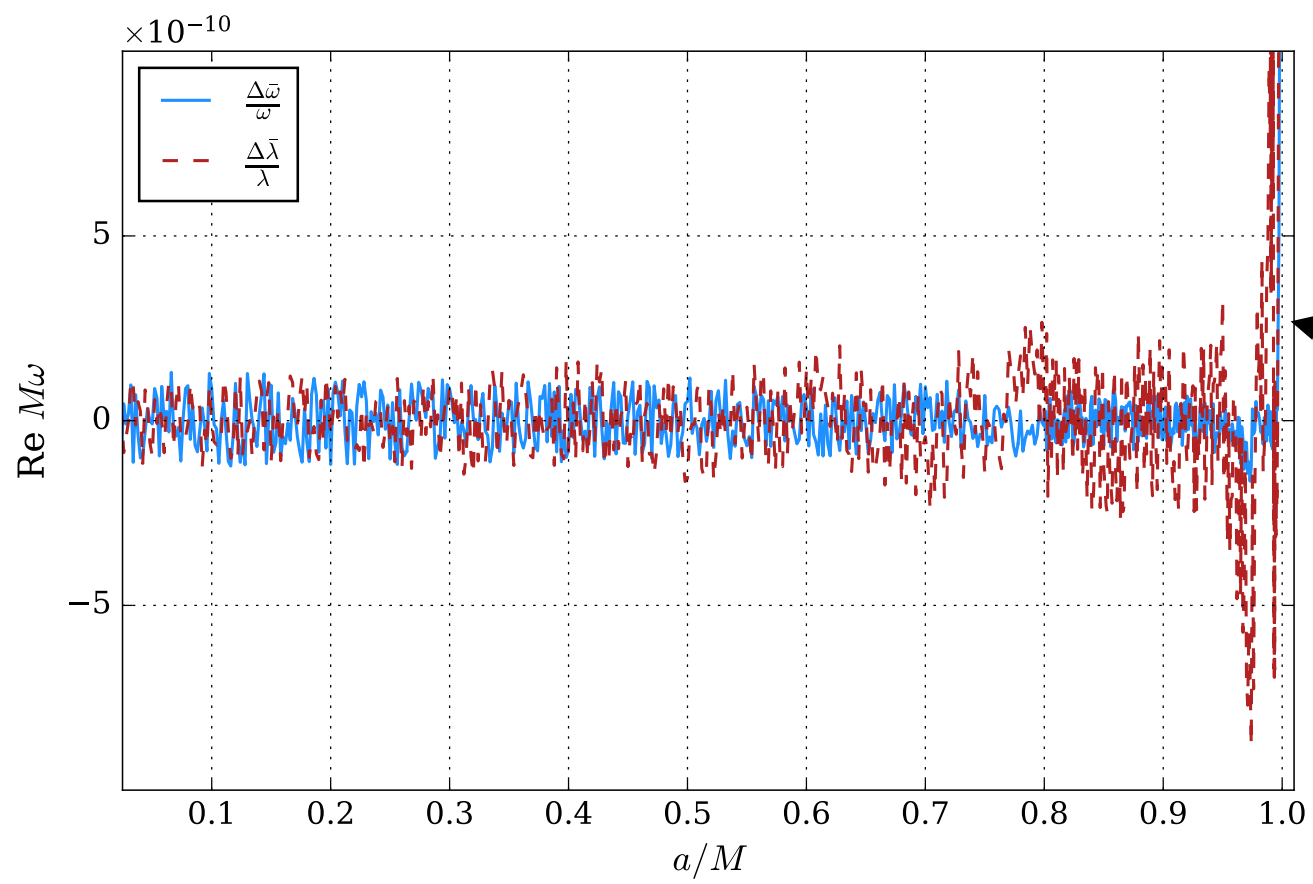
quantization condition for radial equation
involves Painlevé V tau function

Numerics match

$s = -2, \ell = 2, m = 0, 1, 2$



relative error



Zamolodchikov recursion
(continuous fraction method)
fails at extremal point

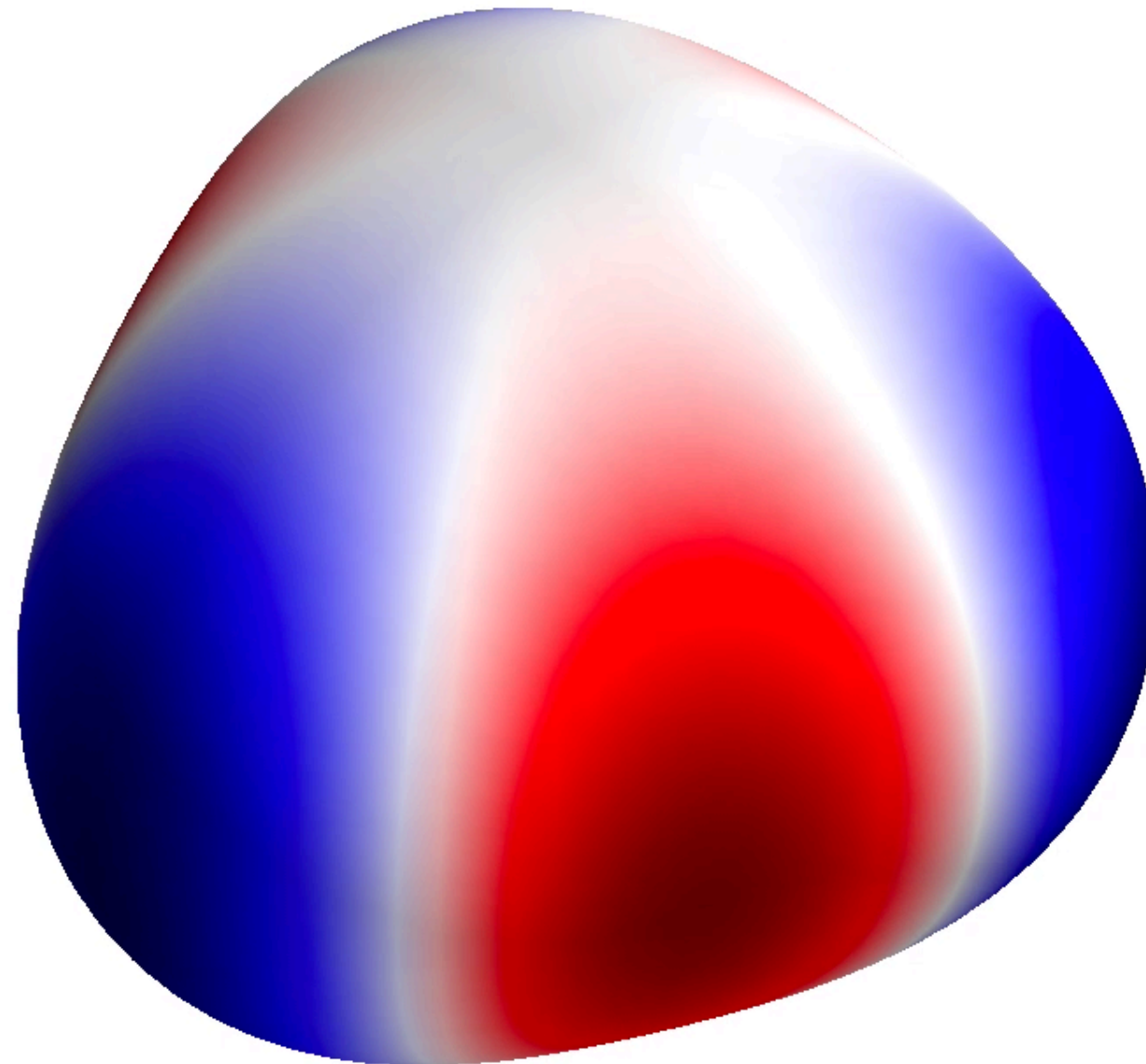
Extremal limit

two types of behavior:

$$\left\{ \begin{array}{ll} M\omega \rightarrow m/2 & \text{co-rotating} \\ M\omega \nrightarrow m/2 & \text{confluent limit} \end{array} \right.$$

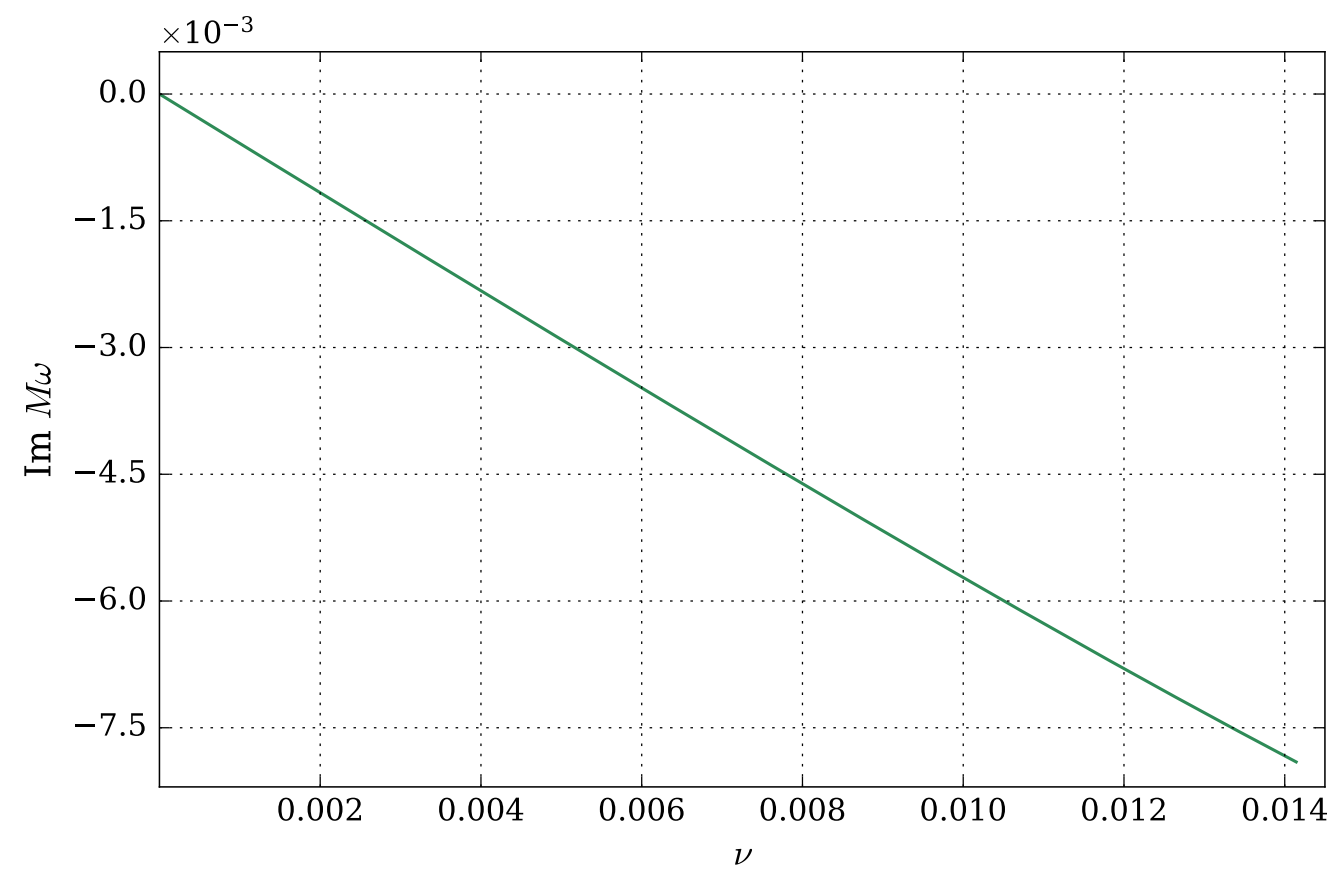
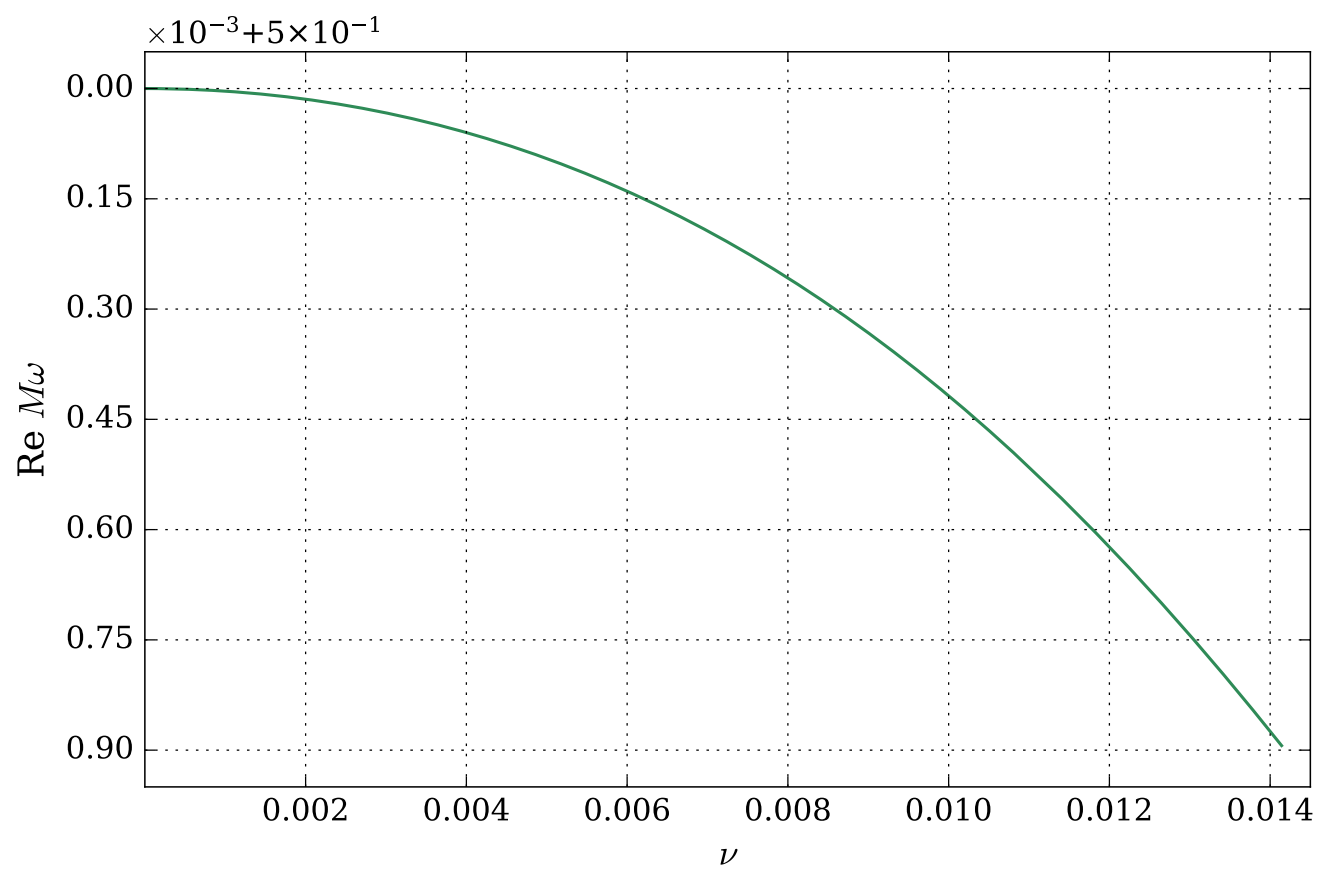
Co-rotating limit:

(most) $m = \ell$ cases



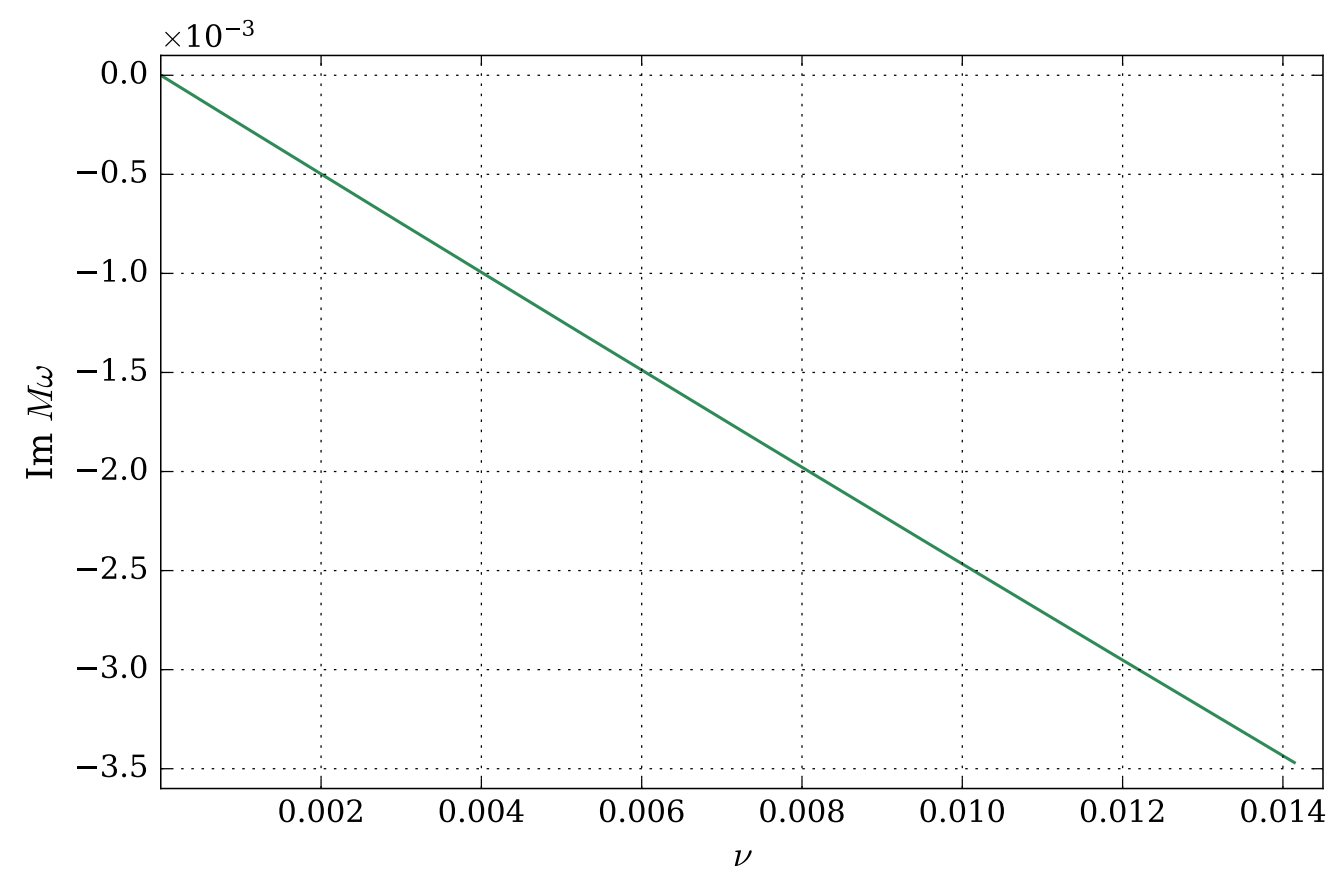
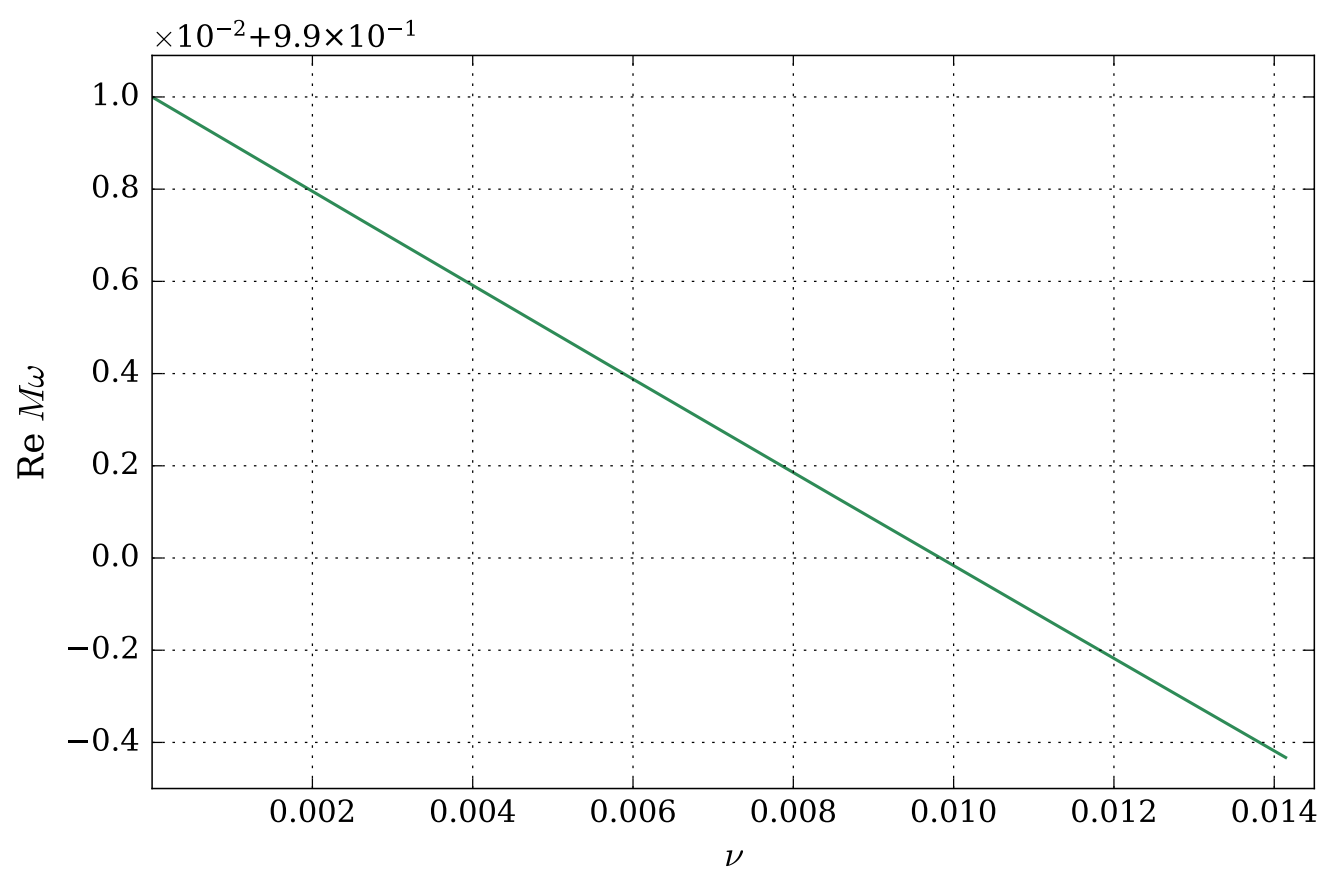
$$M\omega = \frac{m}{2} + \beta_1\nu + \beta_2\nu^2 + \dots, \quad \sigma = 1 + \alpha_0 + \alpha_1\nu + \alpha_2\nu^2 + \dots$$

$$s = 0, \ell = 1, m = 1$$



real α_0

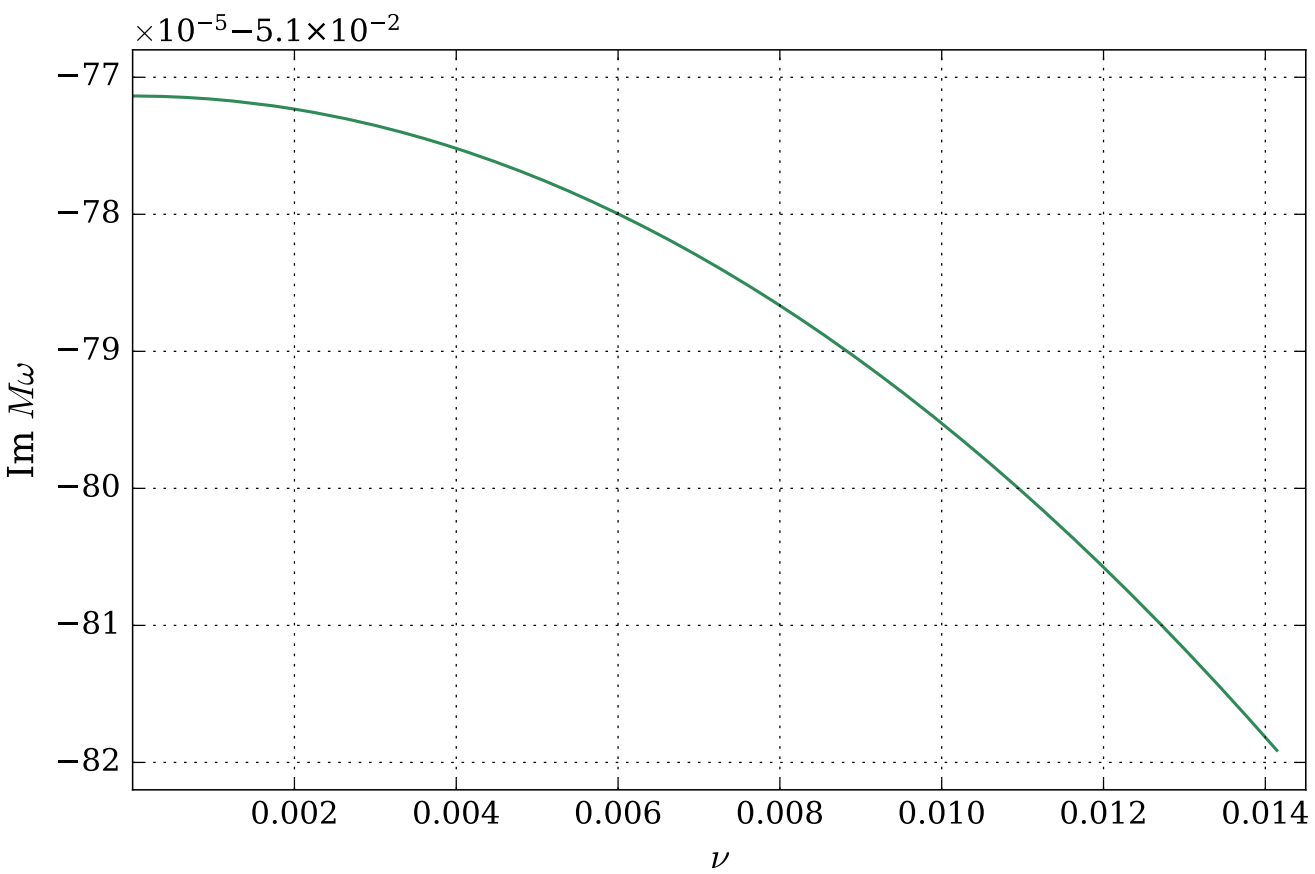
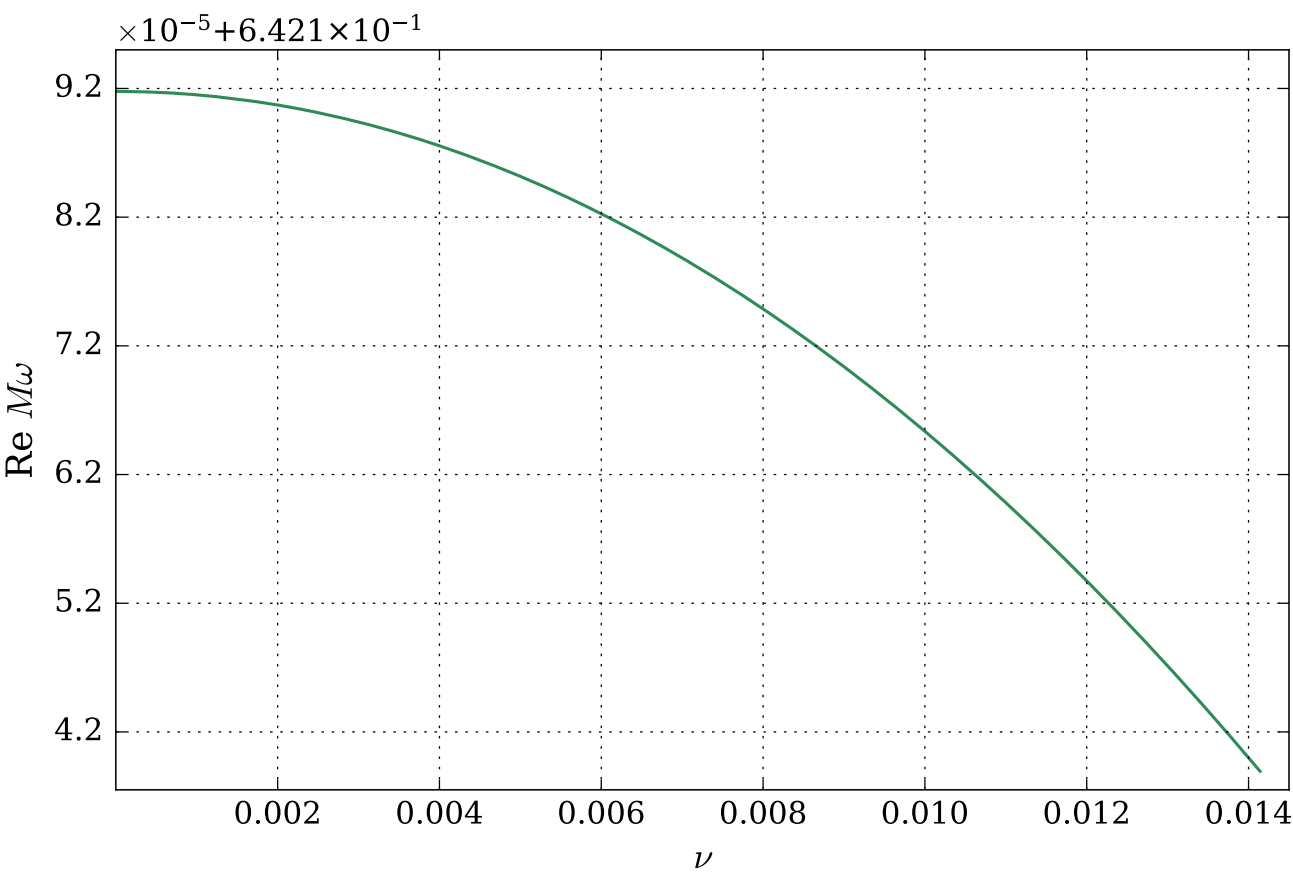
$$s = -2, \ell = 2, m = 2$$



imaginary α_0

Confluent limit

$$s = -1, \ell = 2, m = 1$$



Limit given by Painlevé III tau function

$$\tau_{III}(\vec{\theta}_{\text{ext}}; \sigma, \eta; z_{\text{ext}}) = 0, \quad z_{\text{ext}} \frac{d}{dt} \log \tau_{III}(\vec{\theta}_{\text{ext},-}; \sigma - 1, \eta; z_{\text{ext}}) - \frac{(\theta_{\text{ext},\circ} - 1)^2}{2} = z_{\text{ext}} c_{\text{ext},z}$$

The Reissner-Nordström black hole

$$s = 0, \frac{1}{2}$$

$$\Delta^{-s} \frac{d}{dr} \left(\Delta^{1+s} \frac{dR_s(r)}{dr} \right) + \left(\frac{K(r)^2 - 2is(r-M)K(r)}{\Delta} + 4is\omega r - 2isqQ - {}_s\lambda_\ell \right) R_s(r) = 0$$

$$K(r) = \omega r^2 - qQr$$

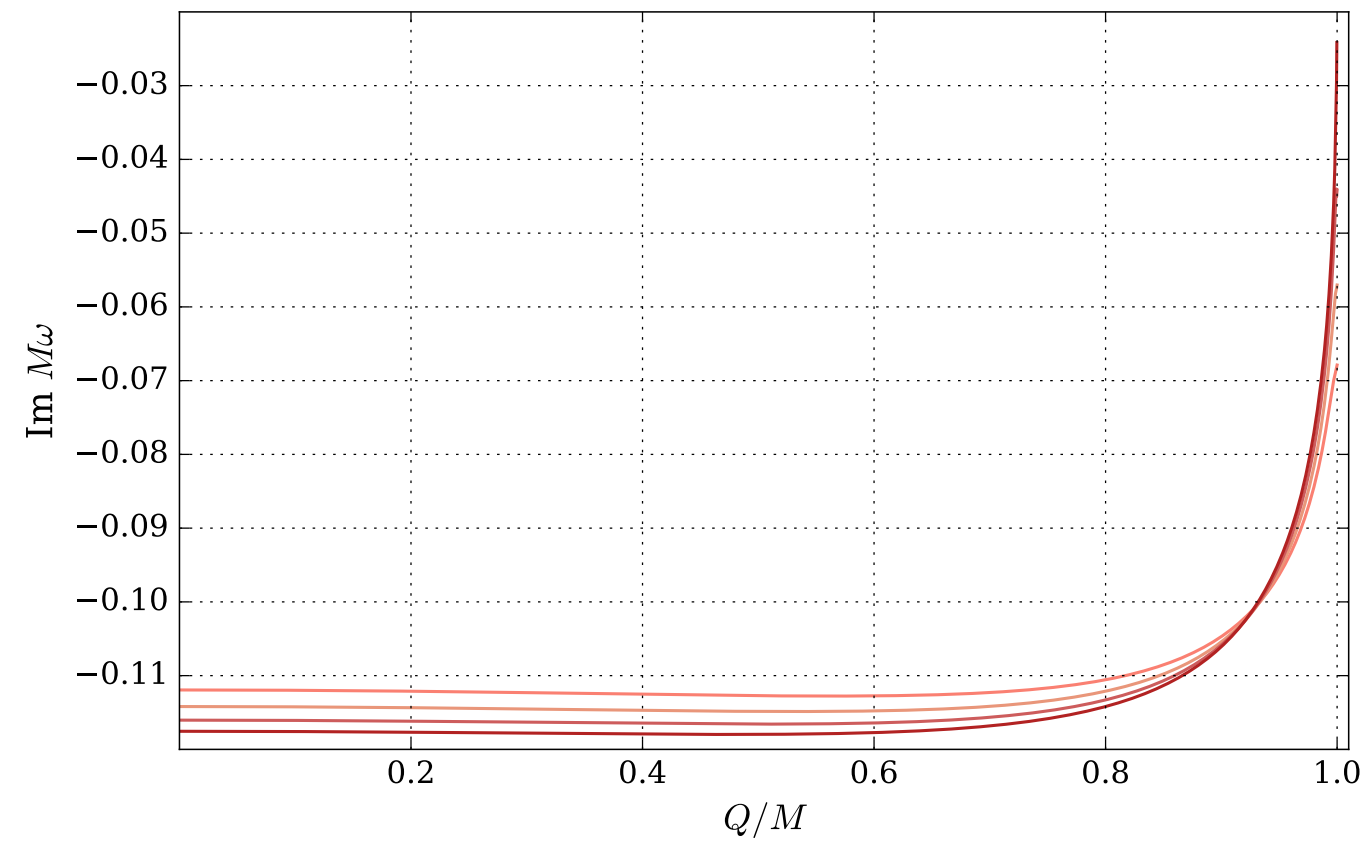
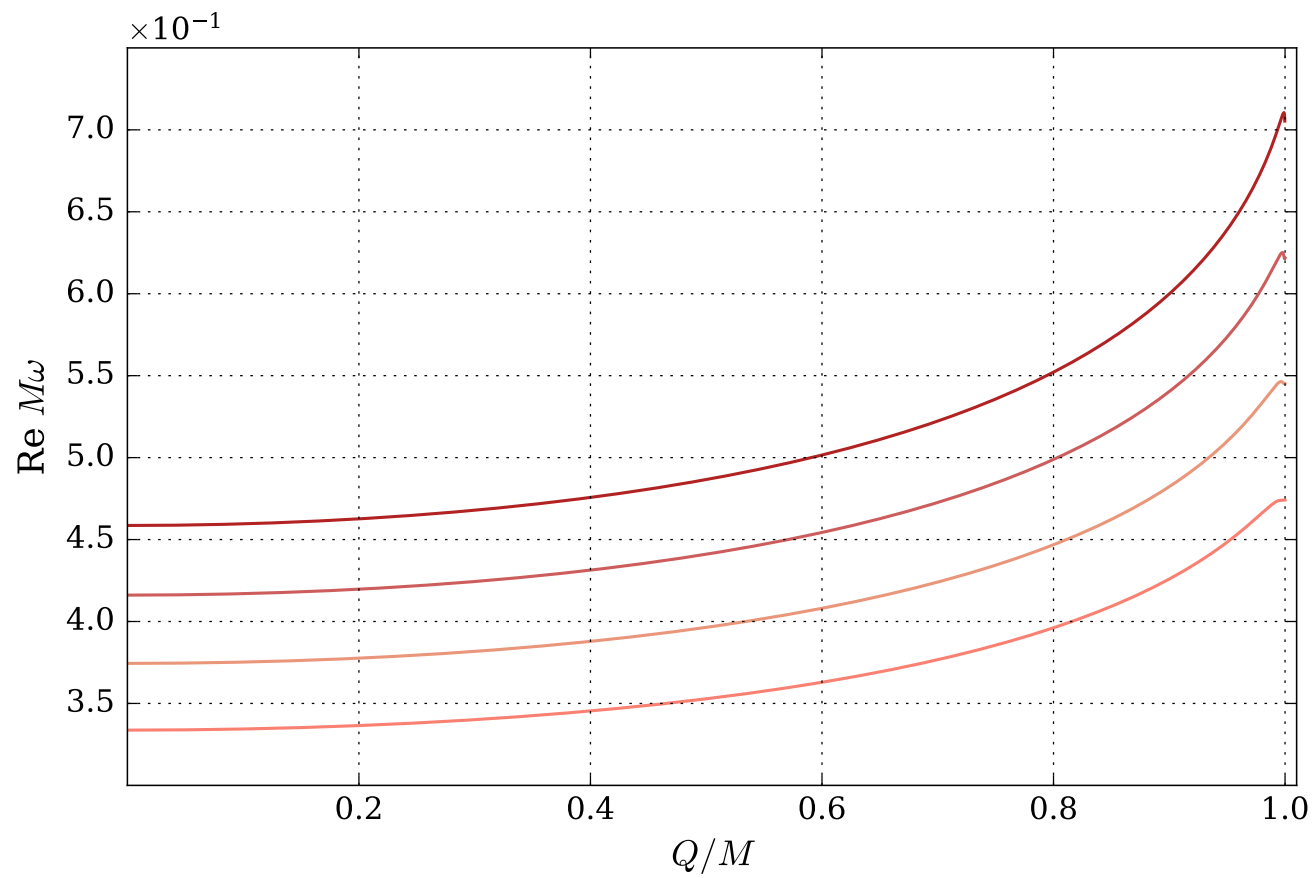
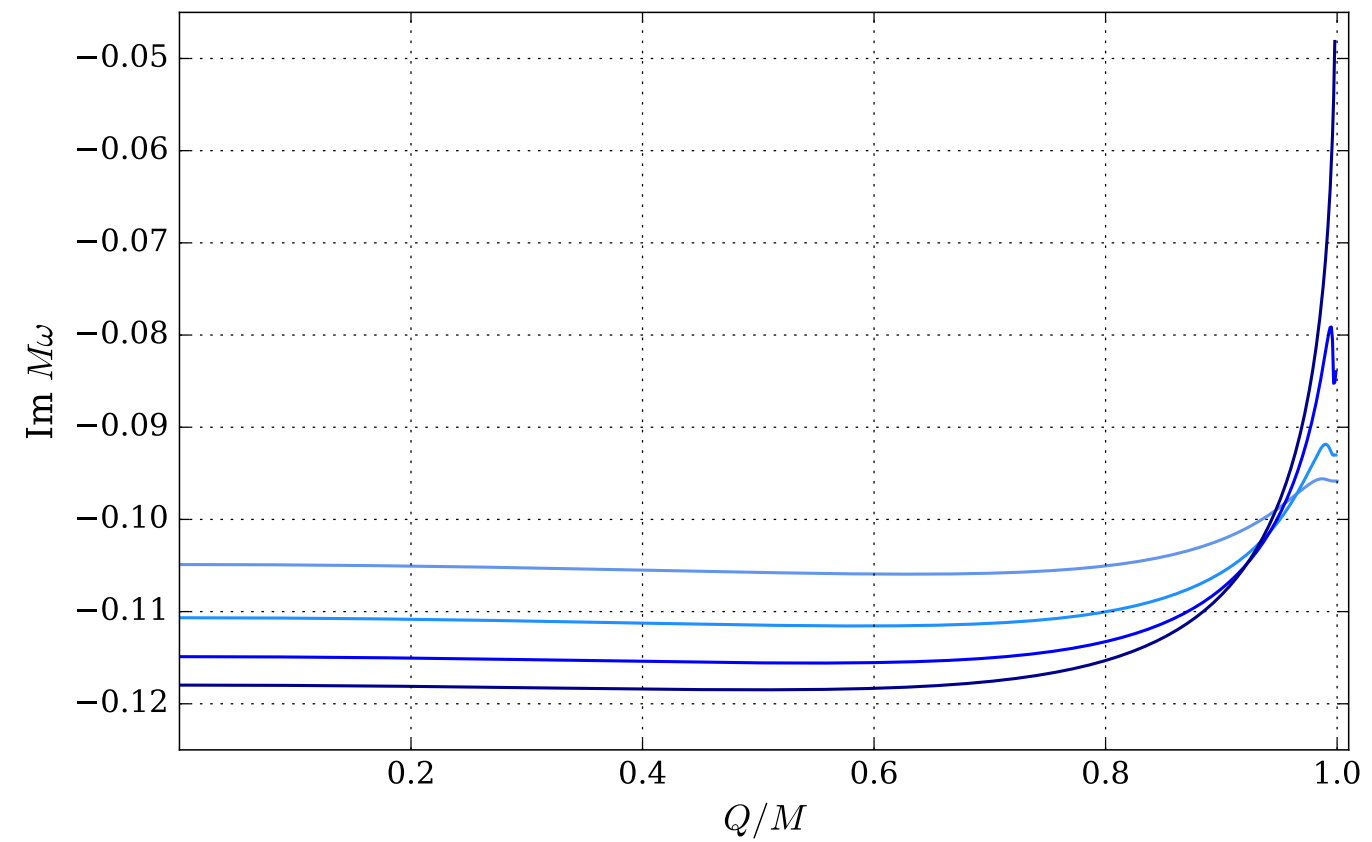
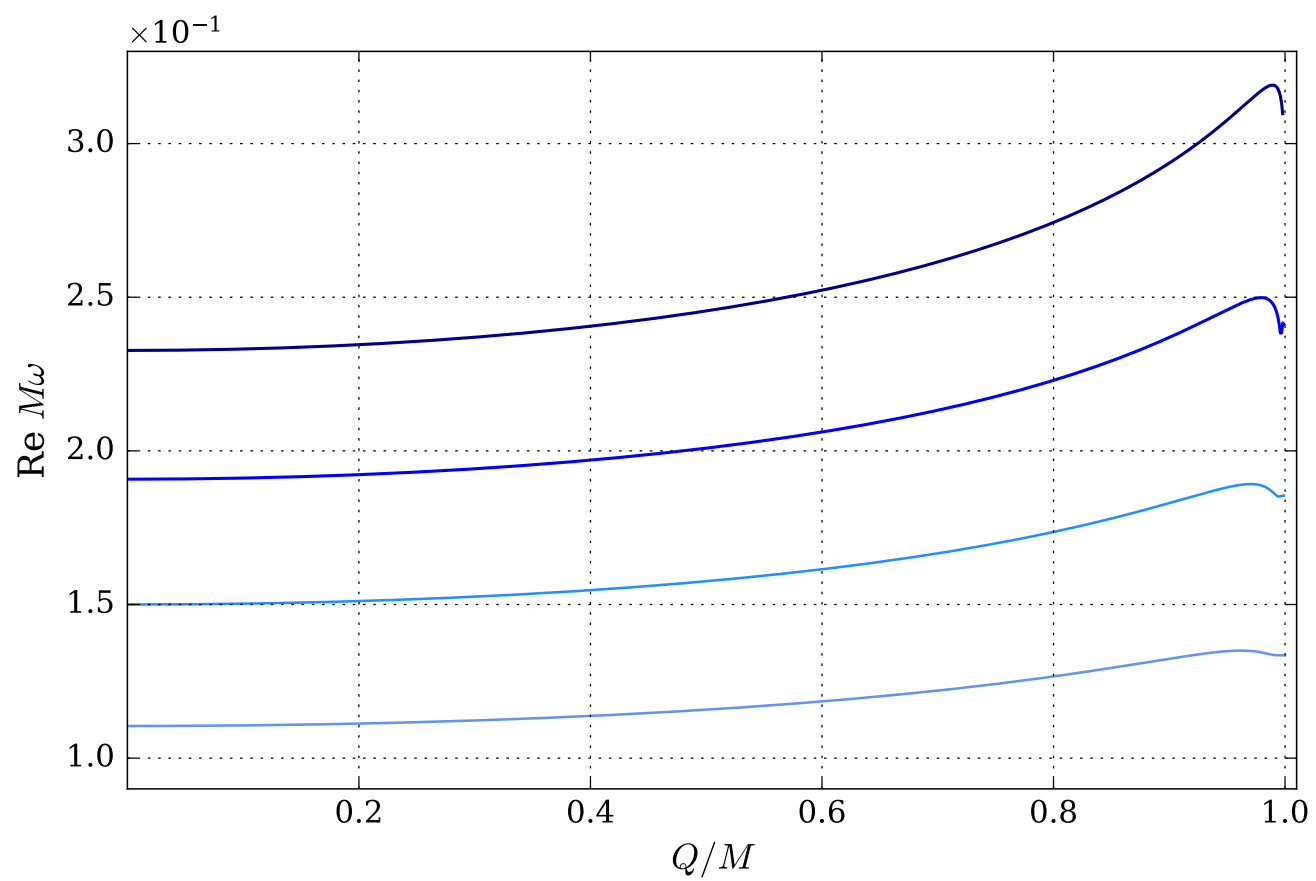
$${}_s\lambda_{\ell,m} = (s - \ell)(s + \ell + 1)$$

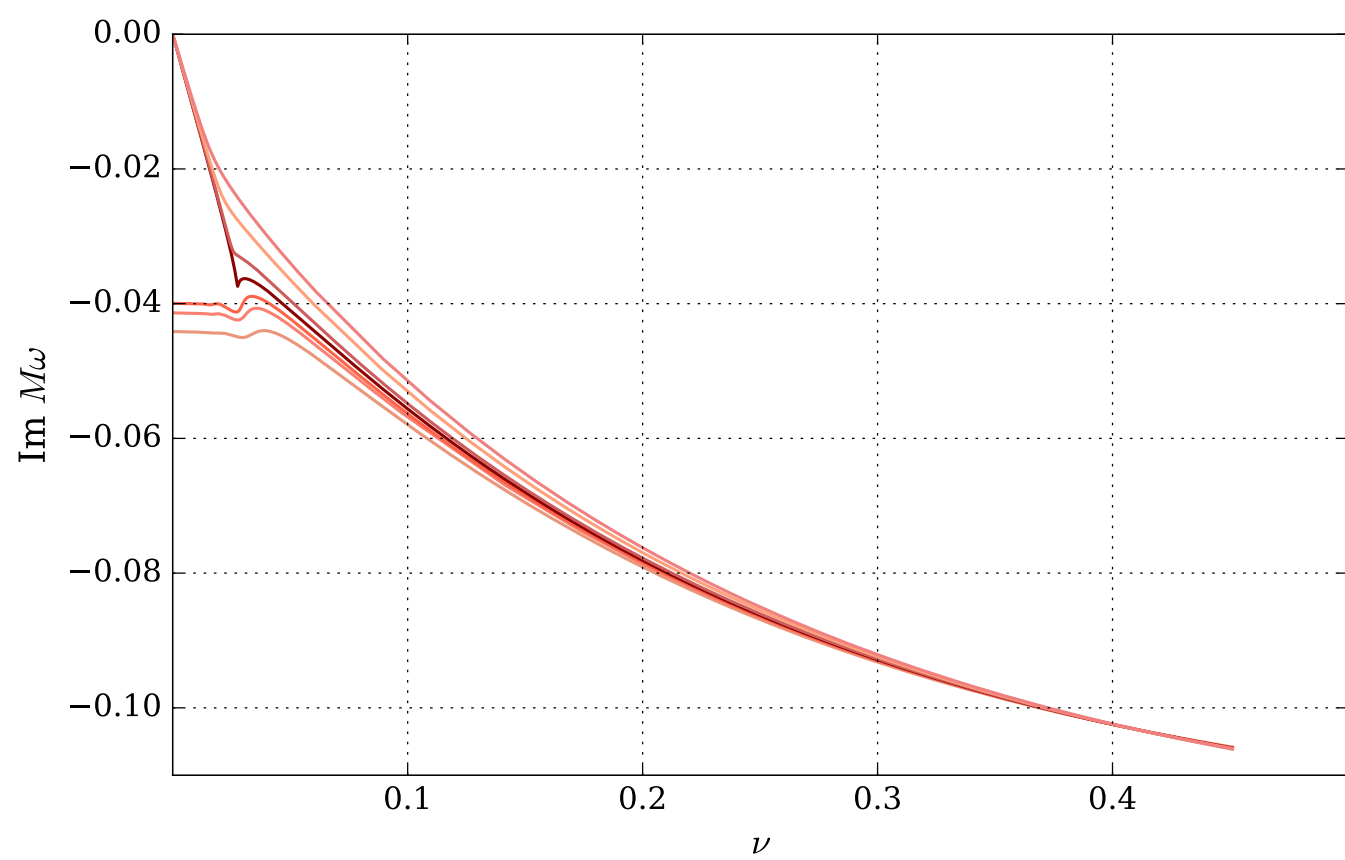
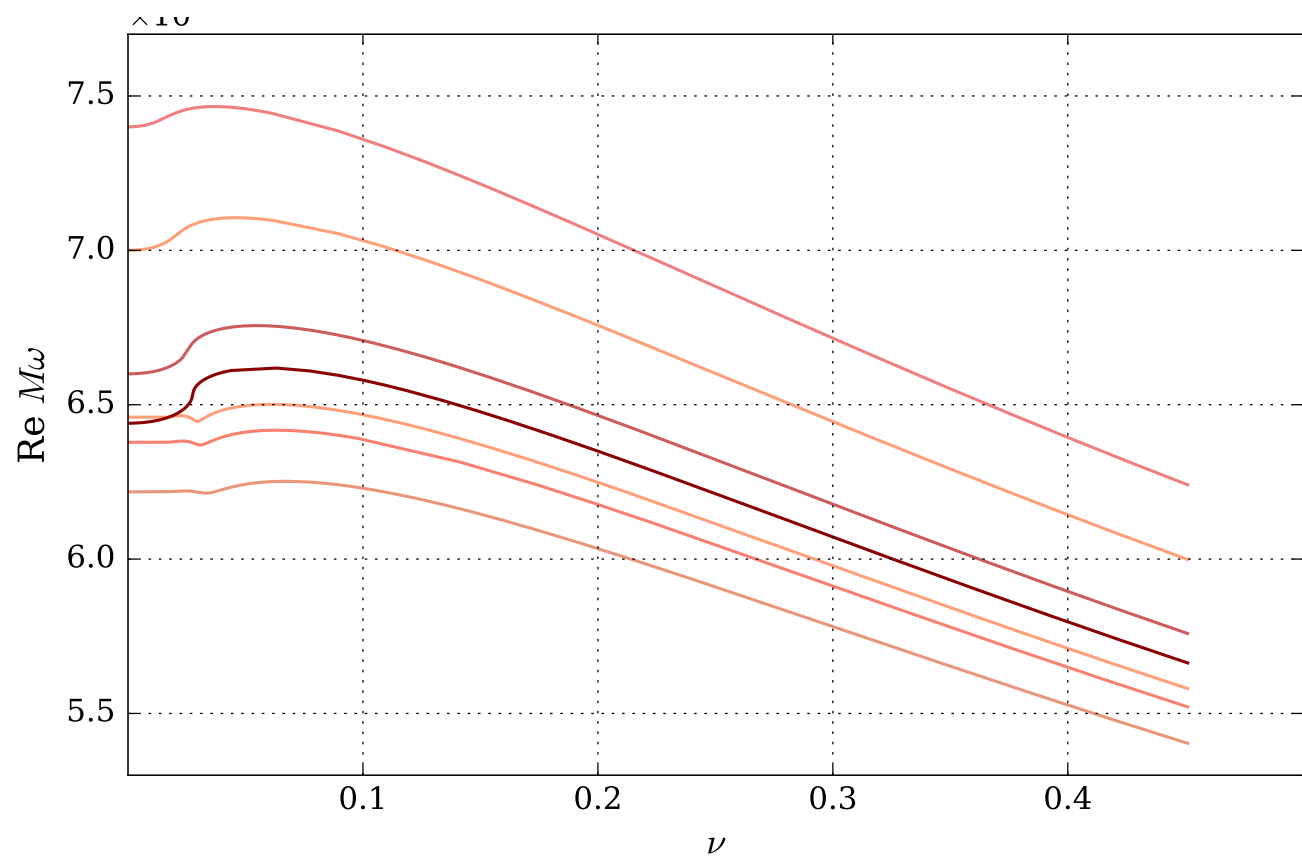
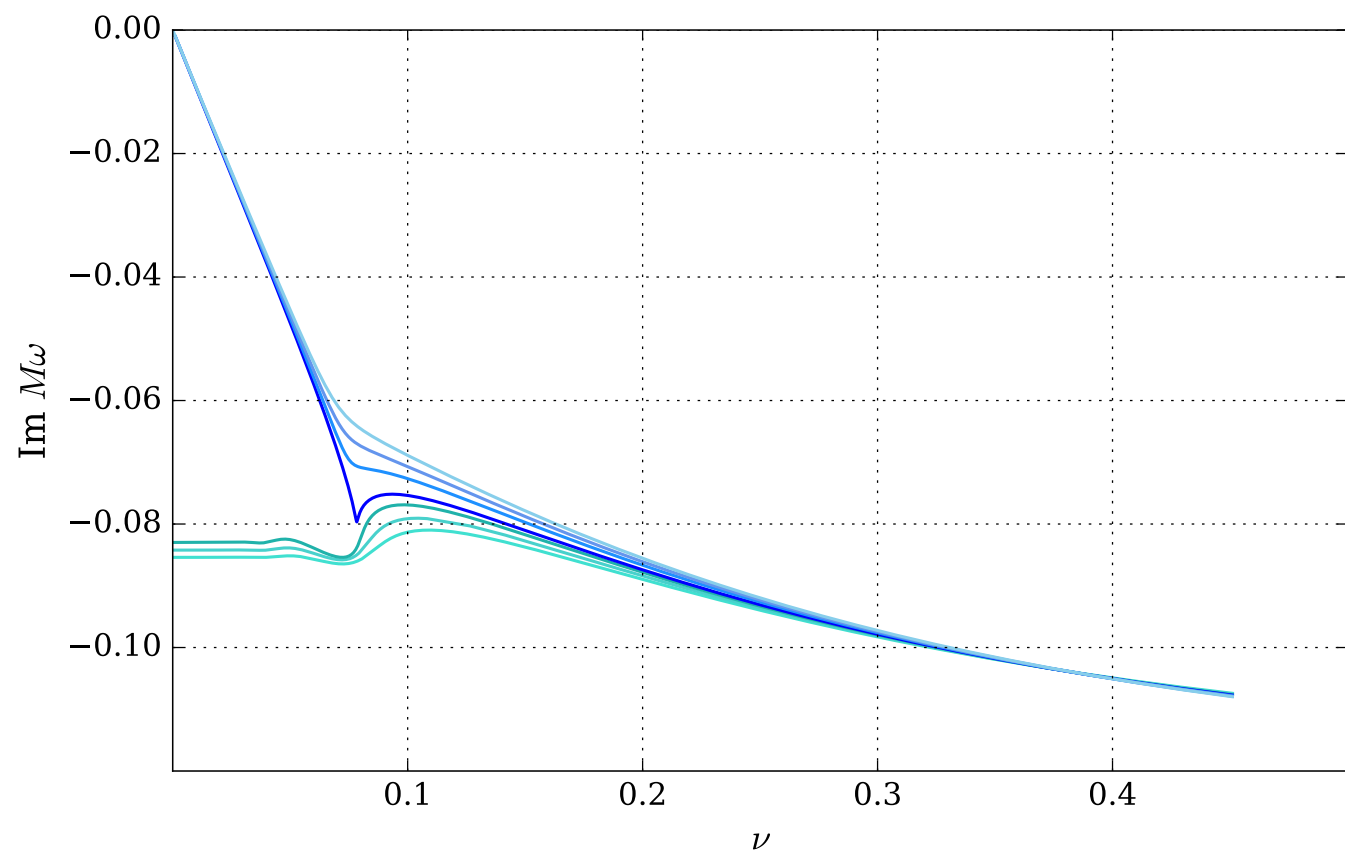
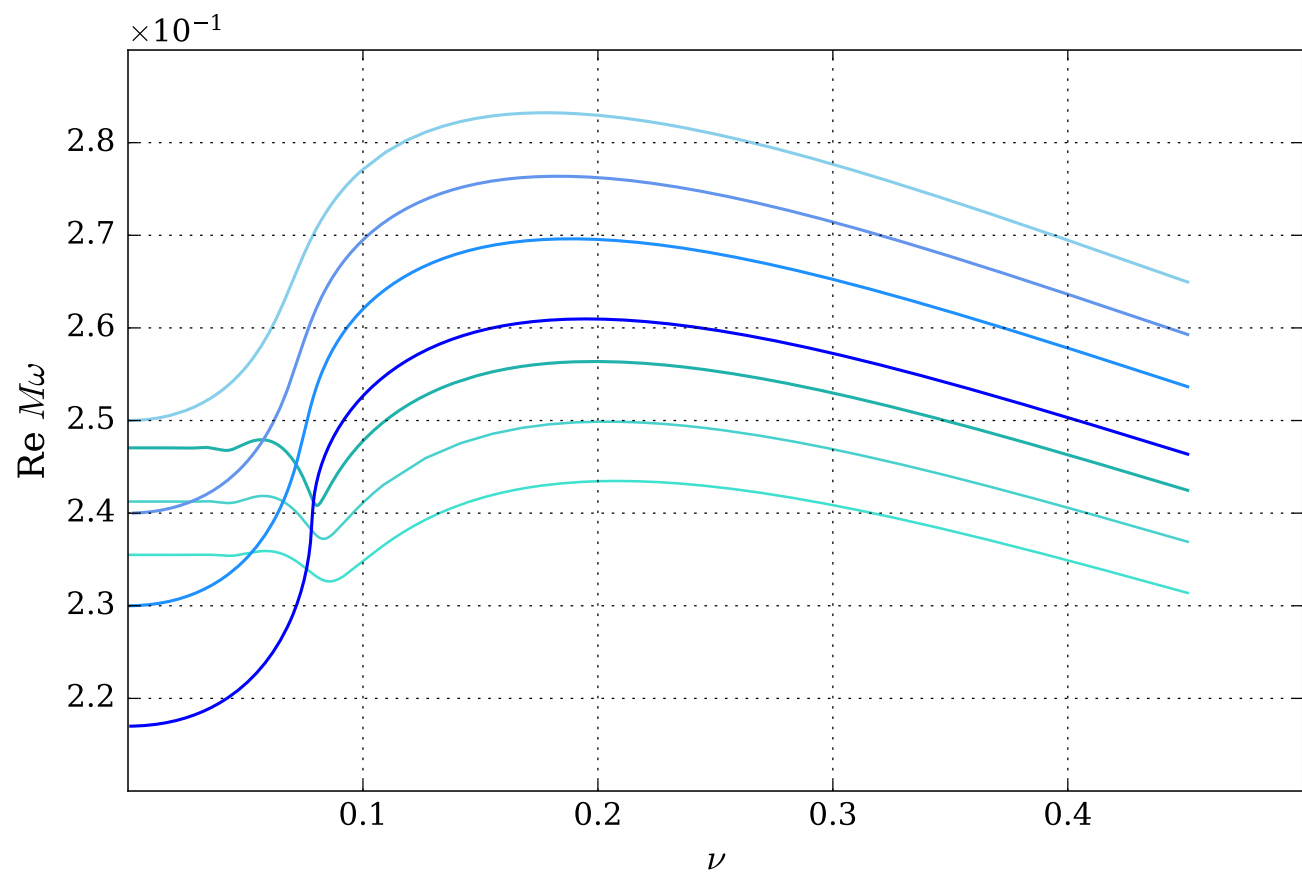
$$\theta_- = s + \frac{i}{2\pi T_-} \left(\omega - \frac{qQ}{r_-} \right), \quad \theta_+ = s + \frac{i}{2\pi T_+} \left(\omega - \frac{qQ}{r_+} \right), \quad \theta_\star = -2s + 2i(2M\omega - qQ),$$

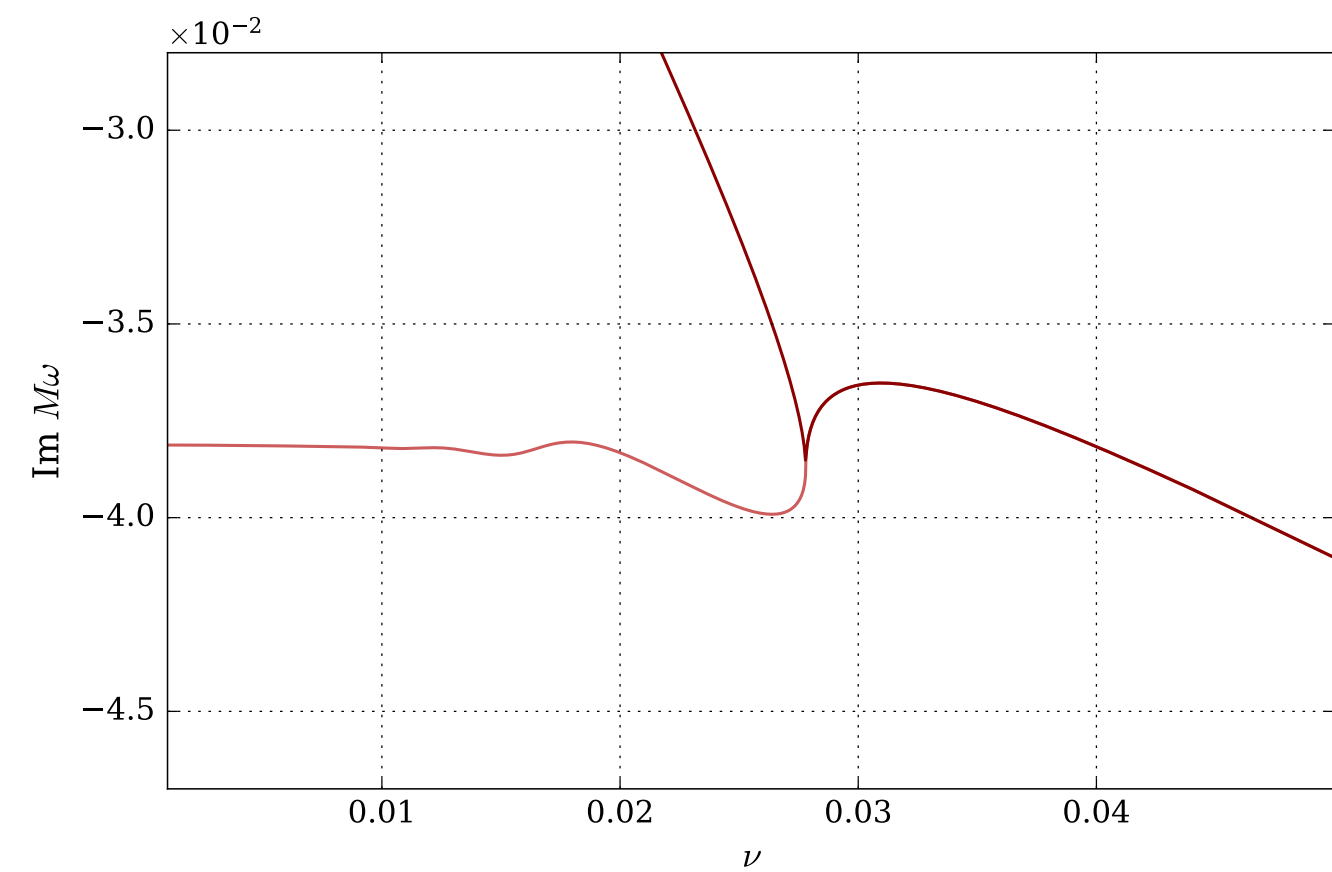
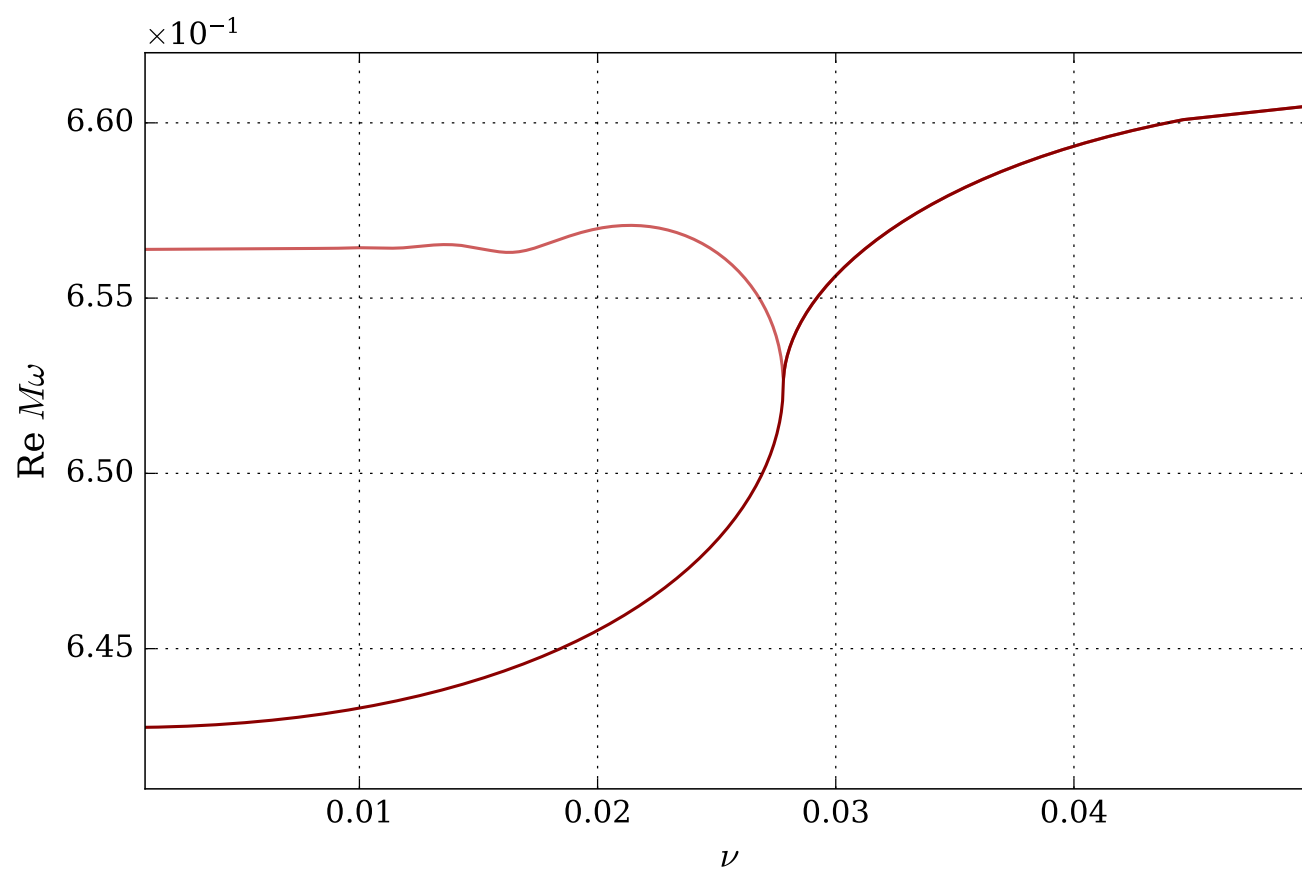
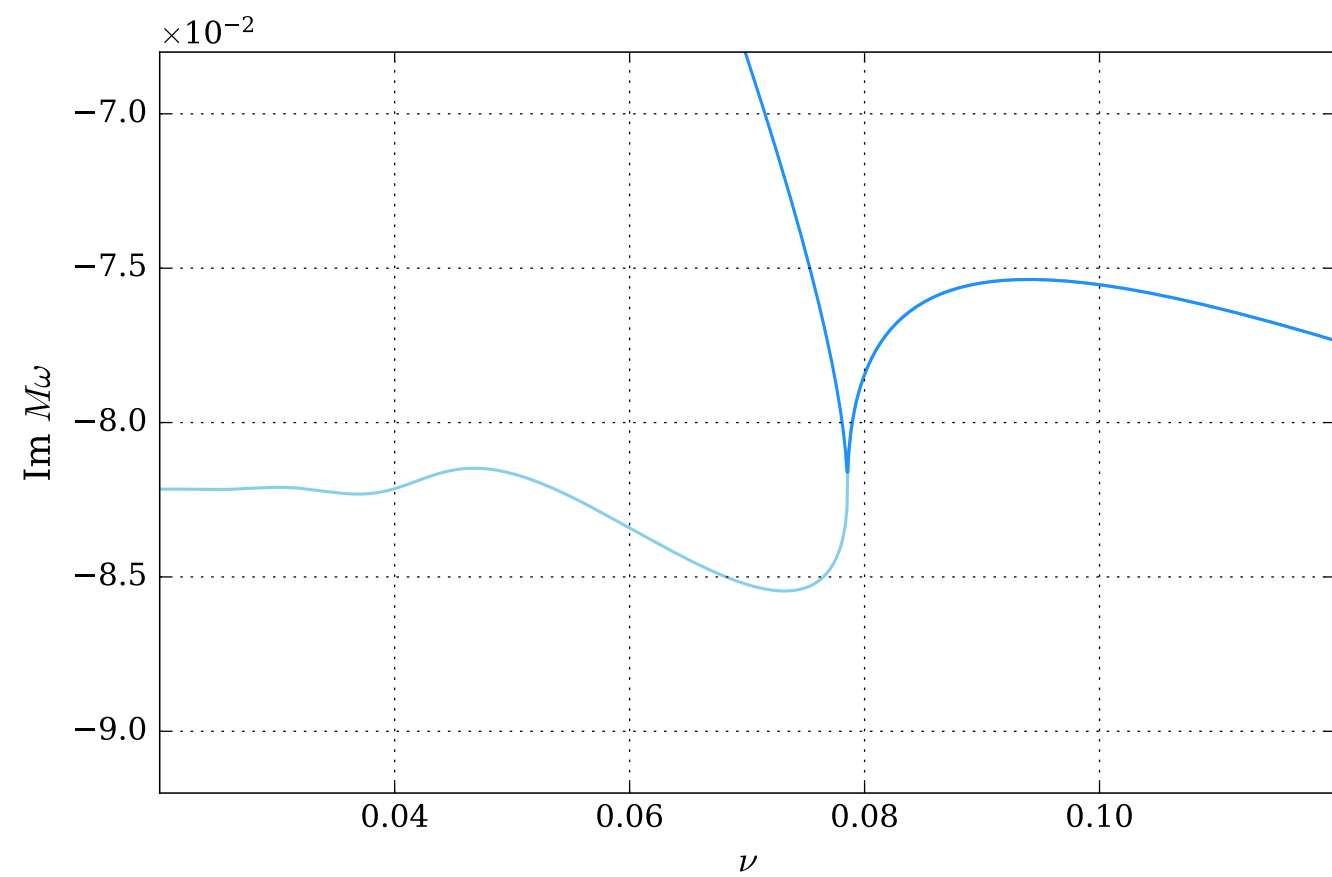
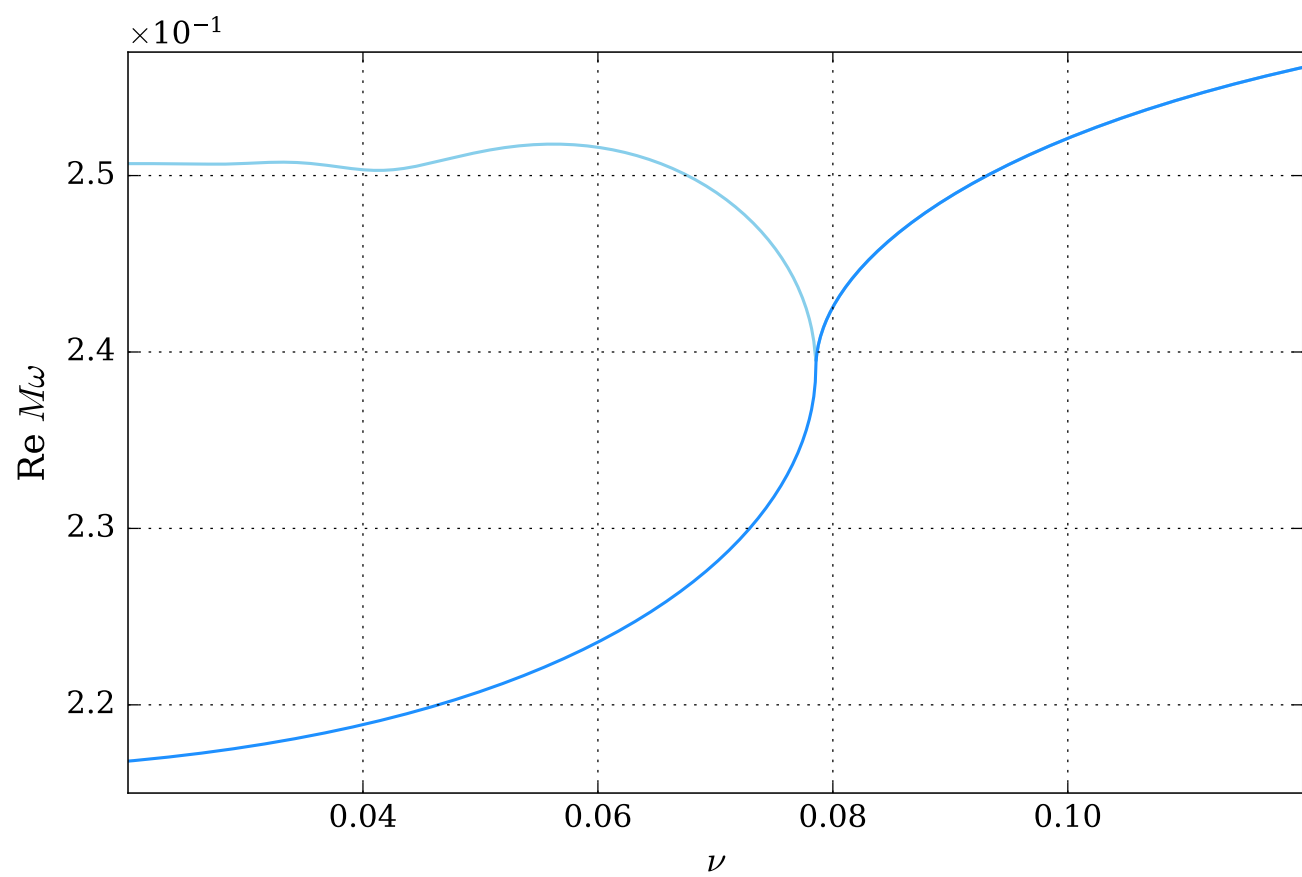
$$2\pi T_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}, \quad r_\pm = M \pm \sqrt{M^2 - Q^2},$$

$$z_0 c_{z_0} = {}_s\lambda_{l,m} + 2s - i(1 - 2s)qQ + (2qQ + i(1 - 3s))\omega r_+ + i(1 - s)\omega r_- - 2\omega^2 r_+^2, \quad z_0 = 2i\omega(r_+ - r_-).$$

$$Q/M = \cos \nu$$



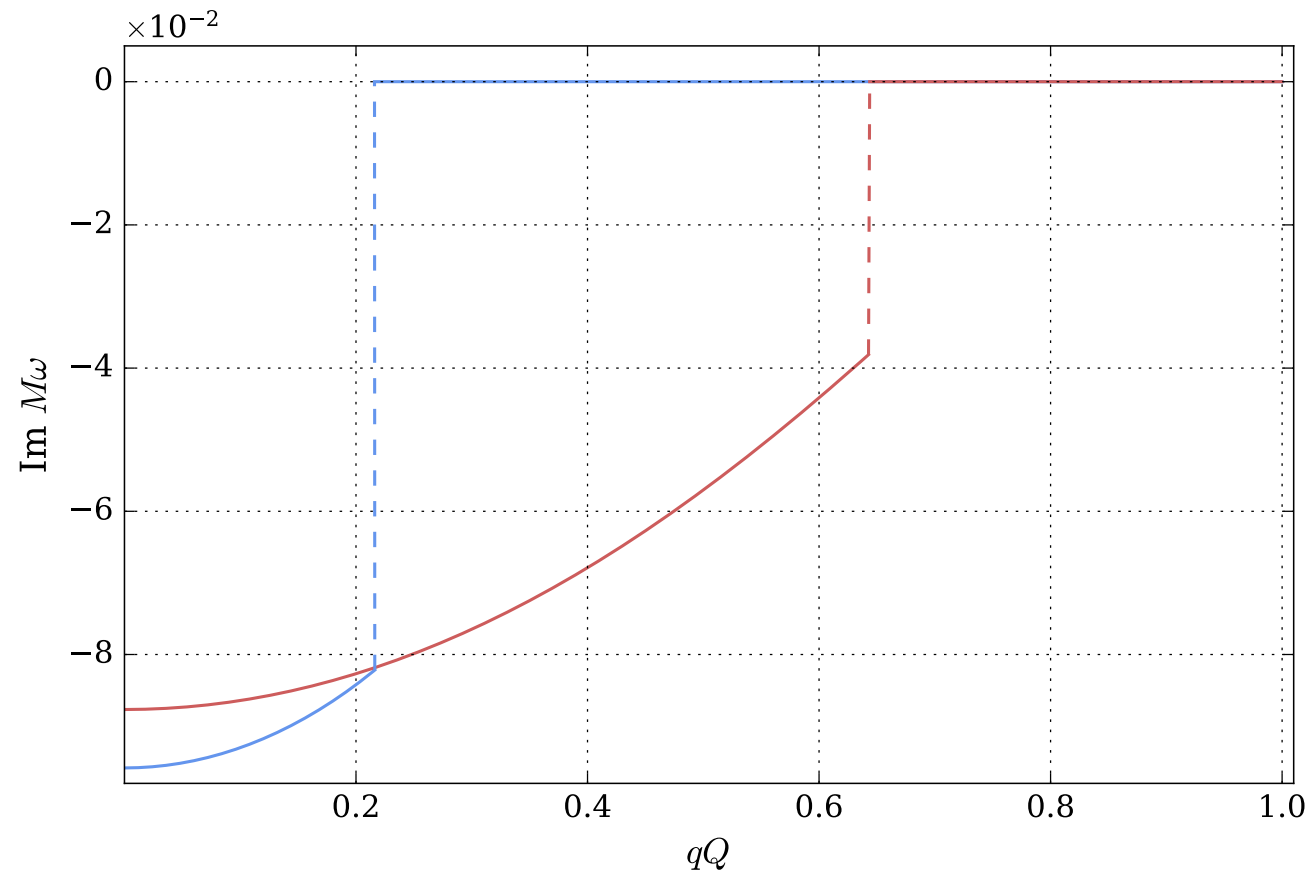
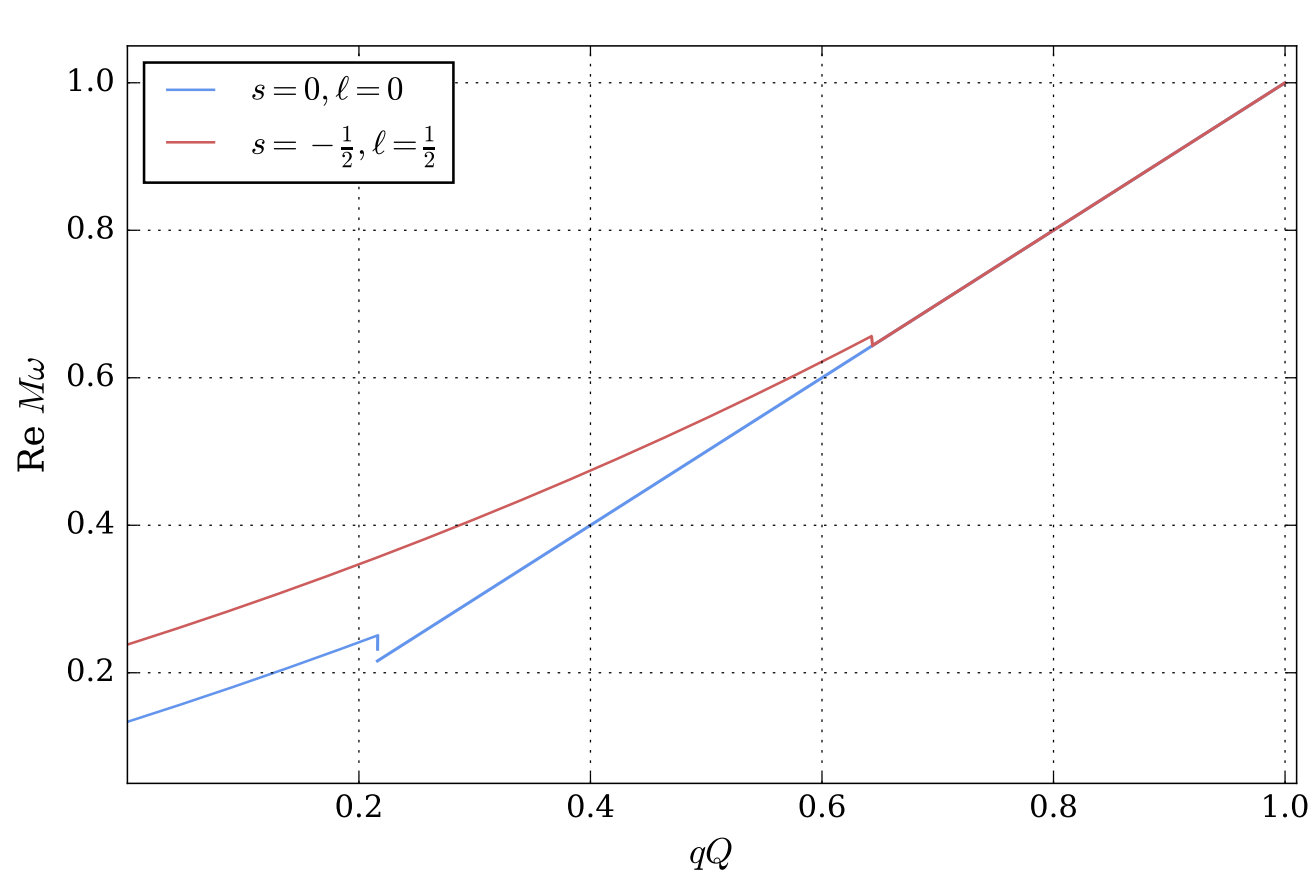




$$qQ_c(s=0) \simeq 0.216228, \quad \text{and}$$

$$qQ_c(s=-1/2) \simeq 0.642745$$

Decoupling at the extremal limit



Still don't know why, but at least know how.

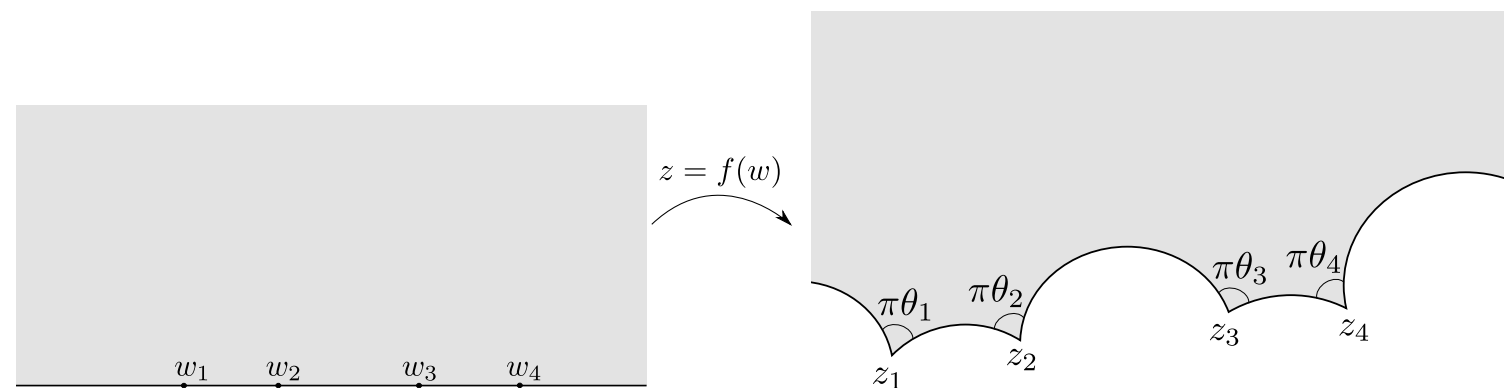
constructive conformal mapping

with T. Anselmo, S. Nejad, R. Nelson and D. Crowdy

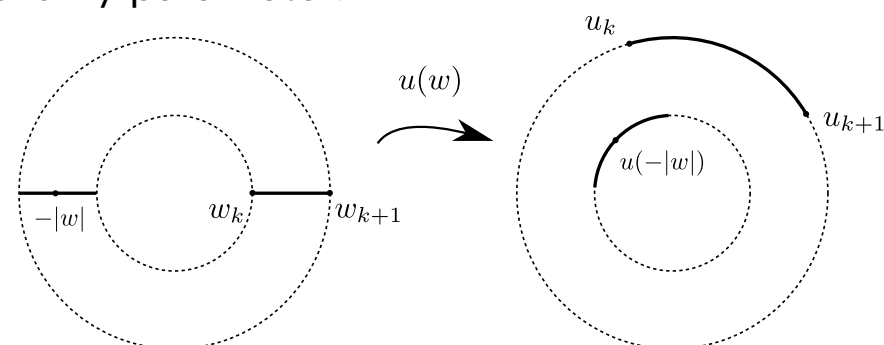
$$\{f(w); w\} \equiv \frac{\partial_w^3 f}{\partial_w f} - \frac{3}{2} \left(\frac{\partial_w^2 f}{\partial_w f} \right) = 2T(w), \quad T(w) = \sum_{k=0}^{n-1} \frac{\alpha_k}{(w - w_k)^2} + \frac{\beta_k}{w - w_k}$$

can be transformed to Fuchsian equation by writing $f(w) = \frac{Y_1(w)}{Y_2(w)}$

solution analytic outside $w = w_k$. can read single monodromy from deficit angle



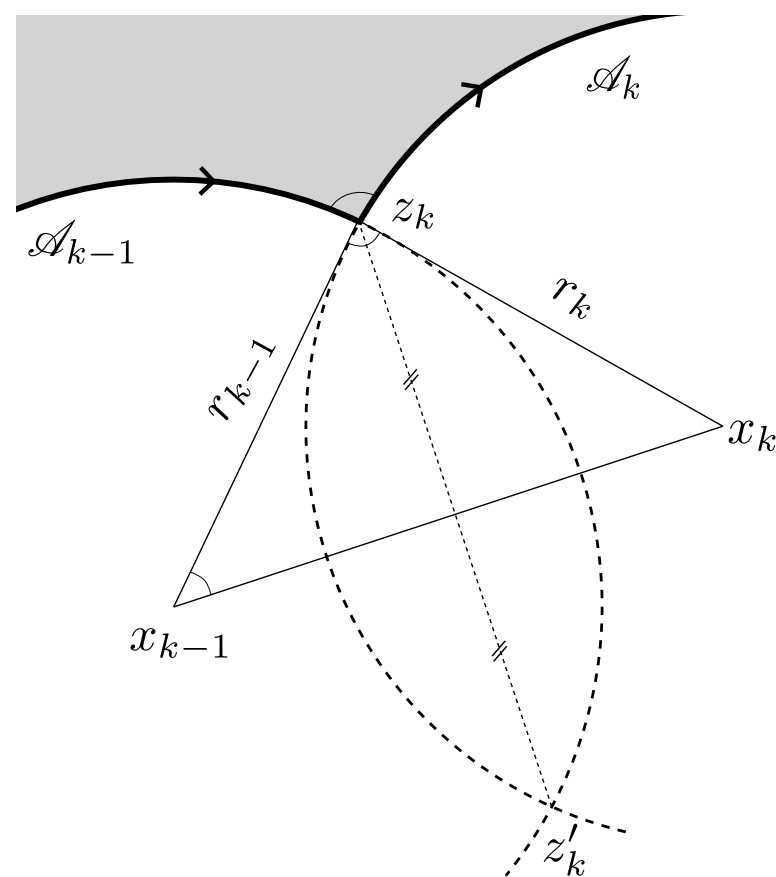
generally complex composite monodromy parameter.



$$\sigma_k = 1 + i\lambda_k$$

Explicit monodromy matrices can be constructed from geometrical representation of polycircular arc domain

$$\mathbf{S}_k = \frac{1}{r_k} \begin{pmatrix} x_k^* & 1 \\ r_k^2 - |x_k|^2 & -x_k \end{pmatrix} \qquad \widehat{\mathbf{M}}_k = \mathbf{S}_k \mathbf{S}_{k-1}^{-1}$$



accessory parameters of $T(w)$ can be obtained by the transcendental equations

$$\tau_{\text{JMU}}(\hat{\rho}_k^+; \{w_k\}) = 0, \qquad \beta_k = -\frac{\partial}{\partial t_k} \log \tau_{\text{JMU}}(\hat{\rho}; \{w_k\}) + \frac{\hat{\theta}_k}{2w_k} + \frac{\hat{\theta}_k}{2(w_k - 1)}.$$

what we learned

- not only a formal solution to the connection problem, but useful and effective way of computing (numerically or analytically);
- advantage to think about monodromy even in usual schemes of calculation (Hill's determinant, continuous fraction);
- any “separable” black hole?
- “factorization” of space-time conformal blocks into 2d chiral ones; CFT interpretation (is it just a trick?);
- pays to consider all monodromy parameters; (extremal) limits better behaved; Relevant physical quantities already translated to monodromy data;

Thank you!