How do we know what the allowed GSO projection rules are?

Choose some rules and then construct correlation fn. via OPE and plumbing fixture.

Check for X-ing symmetry and modular inv.

Passing these $\Rightarrow$ we have a good CFT but not necessarily a good string theory.

Check for the absence of tachyons, absence of vacuum instability etc.

Result: Only 2 consistent string theories in $D=10$: IIA, IIB.

Correlation fn.s in the R-sector can in principle be found using the bosonization rules ($S_\alpha : e^{\frac{i}{2}(\pm\phi_1 \pm \cdots \pm \phi_5)}$

In practice they can be computed from a few OPE & analyticity

$$\psi^\mu(z)\, e^{-\frac{\phi}{2}} S_\alpha(w) \simeq (z-w)^{-1/2} \frac{i}{2} \Gamma^\mu_{\alpha\beta}\, e^{-\frac{\phi}{2}} S^\beta(w)$$

$$\psi^\mu(z)\, e^{-\frac{\phi}{2}} S^\alpha(w) \simeq (z-w)^{-1/2} \frac{i}{2} (\Gamma^\mu)^{\alpha\beta} e^{-\frac{\phi}{2}} S_\beta(w)$$

$$e^{-\frac{3\phi}{2}} S^\alpha(z)\, e^{-\frac{\phi}{2}} S_\beta(w) \simeq (z-w)^{-2} \delta^\alpha_\beta\, e^{-2\phi}(w)$$

$$e^{-\frac{\phi}{2}} S_\alpha(z)\, e^{-\frac{\phi}{2}} S_\beta(w) \simeq (z-w)^{-1}\, i\, \Gamma^\mu_{\alpha\beta}\, e^{-\phi}\psi_\mu(w)$$

+ anti-hol.

Γ^μ's are 16×16 symmetric matrices satisfying: $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} I_{16}$.

$\hookrightarrow$ identity.

In the product to two Γ^μ's, upper spinor index is contracted with lower spinor index.

$$(\Gamma^\mu \Gamma^\nu)^\alpha{}_\beta = (\Gamma^\mu)^{\alpha\gamma}\, \Gamma^\nu_{\gamma\beta}.$$

Ex. $e^{-\frac{\phi}{2}} S_\alpha$, $e^{-\frac{3\phi}{2}} S^\beta$ have conformal weights $(0, 1)$.

Off-shell state in string theory: GSO even states $|V\rangle$ in the CFT satisfying:

① $L_0^-|V\rangle = 0$, $b_0^-|V\rangle = 0$, $L_0^\pm = L_0 \pm \bar{L}_0$, $b_0^\pm = b_0 \pm \bar{b}_0$

② Picture no. -1 in NS sector
$-1/2$ in R sector.

– both in the hol. and anti-hol. sector.

Physical (on-shell) state: $Q_B|V\rangle = 0$, total gh. no. 2

$|V\rangle$ and $|\tilde{V}\rangle$ are equivalent if $|\tilde{V}\rangle - |V\rangle = Q_B|\Lambda\rangle$ for some $|\Lambda\rangle$ satisfying $b_0^-|\Lambda\rangle = 0$, $L_0^-|\Lambda\rangle = 0$.

Examples of physical states:

NSNS: Ex. check that the following state is physical:

$$\zeta_{\mu\nu}\, c\, \bar{c}\, e^{-\phi} e^{-\bar{\phi}} \psi^{\mu} \bar{\psi}^{\nu} e^{ik\cdot X}(0)\,|0\rangle$$

$$k^2 = 0, \quad k^{\mu}\zeta_{\mu\nu} = 0 = k^{\nu}\zeta_{\mu\nu}.$$

graviton, dilaton, 2-form fields.

$$\zeta_{\mu\nu} \equiv \zeta_{\mu\nu} + k_{\mu}\zeta_{\nu} + k_{\nu}\bar{\zeta}_{\mu}, \quad k\cdot\zeta = 0, \quad k\cdot\bar{\zeta} = 0.$$

whiteboardApp.org

Ex. 2. Check that there are no tachyons.

$c\bar{c}\, e^{ik\cdot X}(0)|0\rangle \to$ is not present since it has picture no. $(0,0)$.

A physical state in the NSR sector:

$$\xi^{\mu}_{\alpha}\, c\bar{c}\, e^{-\frac{\phi}{2}} S_{\alpha}\, e^{-\bar{\phi}}\, \bar{\psi}^{\mu}\, e^{ik\cdot X}(0)|0\rangle$$

Ex. $k^2=0, \quad k_\mu \xi^{\mu}_{\alpha}=0, \quad k_\nu \Gamma^{\nu}_{\alpha\beta}\, \xi^{\beta}_{\mu}=0 \quad \left(Q_B|V\rangle = 0\right)$

$\Downarrow$

A spin $\frac{3}{2}$ and spin $\frac{1}{2}$ state

- gravitino + dilatino

Another set of massless spin 3/2 + spin 1/2 state from the RNS.

- same chirality in IIB.
- opposite " in IIA

these theories describe $N=2$ supersymmetric theories in $D=10$.

Type IIB / type IIA supergravity coupled to an infinite tower of massive states. $\Rightarrow$ No UV divergences

Heterotic string theory

Holomorphic sector: Like superstring

Anti-hol- sector: Like bosonic string

(no $\bar{\beta}, \bar{\gamma}, \bar{\xi}, \bar{\eta}, \bar{\phi}, \bar{\psi}^\mu$).

What about D (No of X^μ's) $\to$ cannot be 10 and 26 at the same time.

We take 10 X^μ's : $X^0, \ldots X^9$.

Add a CFT of central charge $(16, 0)$.

$\Rightarrow \bar{T}^X + \bar{T}_{CFT}$ has central charge $\bar{c} = 26$

CFT of central charge (16,0) are quite restrictive.

~ Only CFT's consistent with X-ing symmetry and modular inv.

$\Rightarrow$ $E_8 \times E_8$ heterotic and SO(32) heterotic string.

CFT's have dim (1,0) virasoro primary operators satisfying

$$\bar{J}^a(\bar{z})\,\bar{J}^b(\bar{w}) \simeq \frac{1}{2(\bar{z}-\bar{w})^2} + i f^{ab}{}_c \frac{1}{\bar{z}-\bar{w}} \bar{J}^c(\bar{w})$$

↳ structure consts. of $E_8 \times E_8$ or SO(32)

$\xi^a_\mu\, c\bar{c}\, e^{-\phi}\psi^\mu \bar{J}^a e^{ik\cdot X}(0)|0\rangle$

$\rightarrow$ phys. if $k^2=0$, $k^\mu \xi^a_\mu = 0$

$\exists$ massless gauge fields.

Heterotic string theory describes $N=1$ supergravity in $D=10$ coupled to $E_8 \times E_8$ or $SO(32)$ super Yang-Mills and ∞ tower of massive states.

For simplicity we'll focus on heterotic string theory $\rightsquigarrow$ involves less writing
– only two sectors NS, R instead of four.

Generalization to type II is straightforward.

Instead of string theory, let us analyze the CFT a bit more.

How to build states with picture no. q?

Begin with $|q,k\rangle = e^{q\phi} e^{ik\cdot X_{(0)}}(0)$

Apply $\beta_n, \gamma_n, b_n, c_n, \tilde{b}_n, \tilde{c}_n$ & matter ops.

$$\beta(z) = \sum \beta_n z^{-n-\frac{3}{2}}, \quad \gamma(z) = \sum_n \gamma_n z^{-n+\frac{1}{2}}$$

$$\varepsilon x.\ \beta(z)\, e^{q\phi}(0) \sim z^{q}, \quad \gamma(z)\, e^{q\phi}(0) \sim z^{-q}$$

$\Rightarrow \varepsilon x.\ \beta_n |q,k\rangle = 0$ for $n \geq -q-\frac{1}{2}$, $\gamma_n |q,k\rangle = 0$ for $n \geq q+\frac{3}{2}$.

whiteboardApp.org

$$\begin{cases} \beta_n |q,k\rangle = 0 & \text{for } n \geq -q-\frac{1}{2} \\ \gamma_n |q,k\rangle = 0 & \text{for } n \geq q+\frac{3}{2}. \end{cases}$$

Ex. n is $\mathbb{Z}+\frac{1}{2}$ for $q \in \mathbb{Z}$
n is $\mathbb{Z}$ for $q \in \mathbb{Z}+\frac{1}{2}$

Unless $\boxed{q = -\frac{1}{2}, -1,}$ $-\frac{3}{2}$, there is at least on $n > 0$ s.t. $\beta_n |q,k\rangle \neq 0$ or $\gamma_n |q,k\rangle \neq 0$.

The spectrum of L_0 is unbounded from below.

Example: $q = 0$. $\beta_n |0,k\rangle = 0$ for $n \geq -\frac{1}{2}$
$\gamma_n |0,k\rangle = 0$ for $n \geq \frac{3}{2}$

$\gamma_{\frac{1}{2}} |0,k\rangle \neq 0$

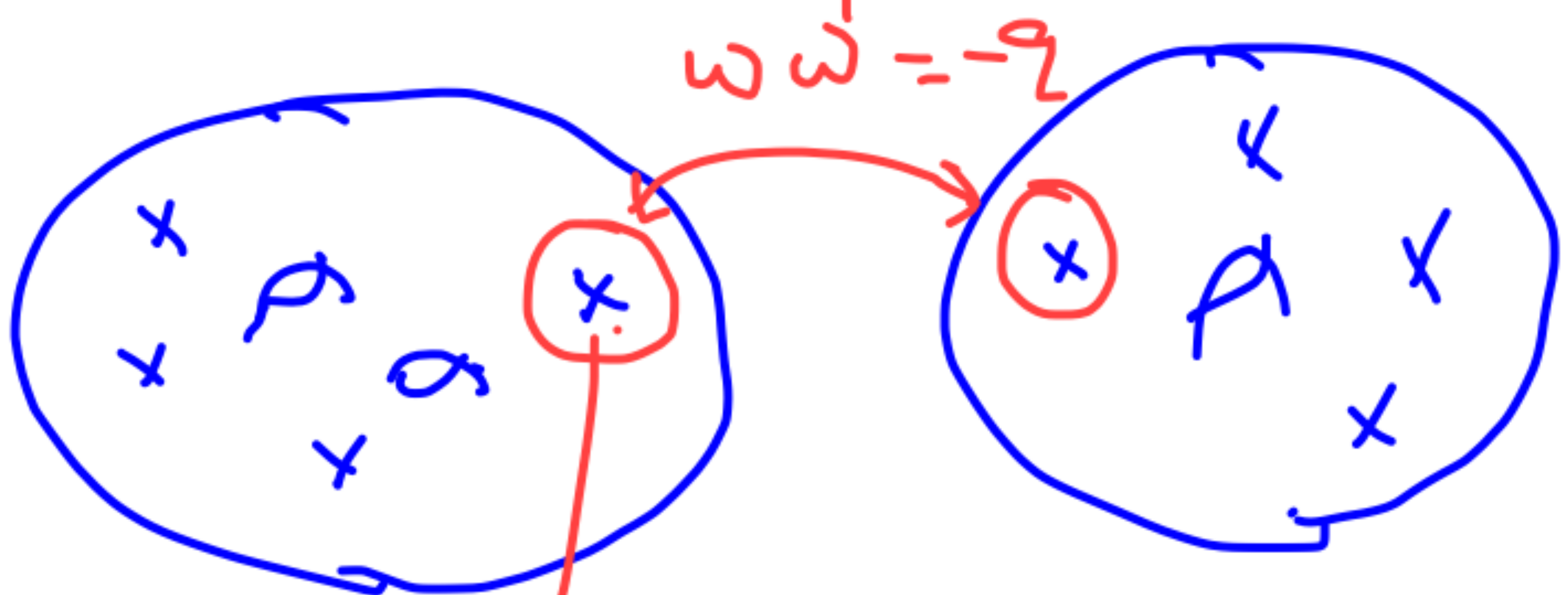
$\underbrace{(\gamma_{1/2})^M |0,k\rangle \neq 0}_{\text{Ex.}}$ → Reduces L_0 e.v. by $\frac{1}{2}M$.

Consider plumbing fixture: $ww' = -q$

$\Downarrow$ insert:

$$\Phi_r(0)\ \Phi_s(0)\ \langle \Phi_r^c \mid \Phi_s^c \rangle\ q^{h_s}\bar{q}^{\bar{h}_s}$$

Picture no. $\quad p \quad -2-p \quad -2-p \quad p$



picture no. of this is determined by the rest of the vertex op & genus.

p is not determined by picture no. conr.

Naive guess: Sum over p

In actual practice: Fix p.

What value of p should we use?

Ans. Any value (formally)

We have unbounded L_0 ev - from below.

q^{h_s} h_s can become arbitrarily -ve.

→ make sense by analytic continuation.

$$1+q^{-1}+q^{-2}+\cdots = \frac{1}{1-q^{-1}} = -\frac{q}{1-q}$$

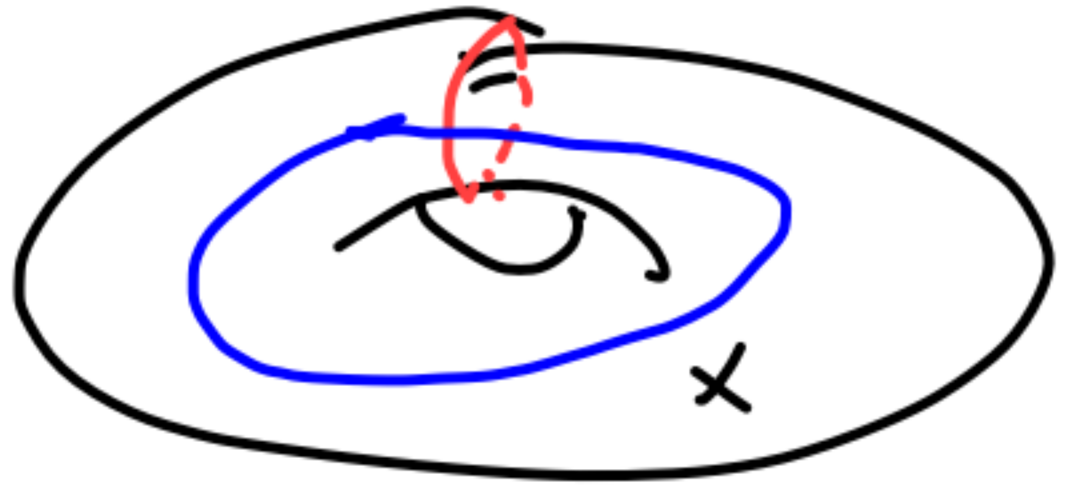
$p=-1$ does not have this problem.

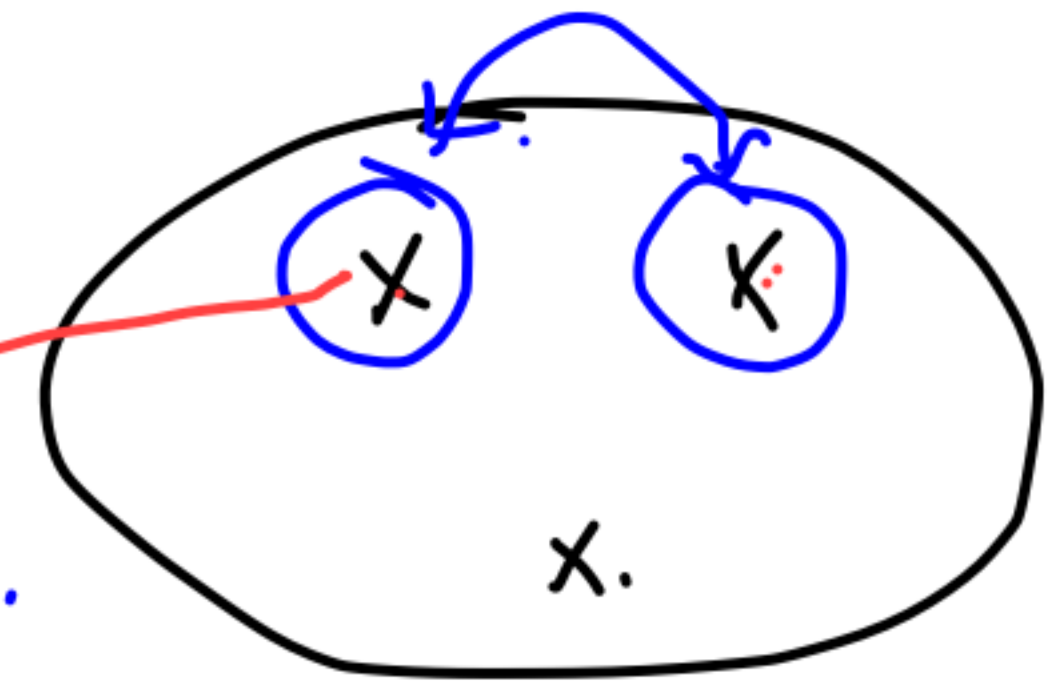
$p=-\frac{1}{2}, -\frac{3}{2}$ have mild problem due to β_0, γ_0

→ cleverly taken care of in string theory

Another point about higher loop amplitudes.
$\rightarrow$ we have to sum over spin structures.
(periodic and anti-periodic b.c. on β, γ, ψ^μ along different non-contractible cycles)

Physical reason
$\rightarrow$ plumbing fixture.





Either NS or R

$\Rightarrow$ anti-periodic or periodic b.c.
along the $|w| = |q|^{1/2}$ circle.

$\mathcal{M}_{g,m,n}$: Moduli space of Riemann surface of genus g with m NS and n R puncture.

$\mathcal{S} \rightarrow$ include sum over spin structure.

$\mathcal{M}_{g,m,n}$

String amplitudes: We need to integrate something over $\mathcal{M}_{g,m,n}$.

$\underbrace{\text{something}}_{?}$

Two possible approaches.

① Make world-sheet SUSY as manifest as possible.

$\rightsquigarrow$ supermoduli space

– besides the usual bosonic coordinates of $M_{g,m,n}$, we have a set of grassmann odd variables.

$\rightarrow$ supermoduli # depends on g, m, n.

(2) Carry out the integration over the grassmann odd variables explicitly.

→ can be done locally in patches of $\mathcal{M}_{g,m,n}$.

⇒ Leads to insertion of PCO's on the Riemann surface $\prod_{\alpha=1}^{K} X(y_\alpha)$

Choice of y_α ⇒ Choice of gauge.

$\alpha : 1, 2, \cdots, K.$

$$K + m\times(-1) + n\times\left(-\tfrac{1}{2}\right) = 2g-2 \Rightarrow \underbrace{K = 2g-2+m+\tfrac{n}{2}}_{\text{\# of odd supermoduli}}$$

Construct the amplitude.

find the analog of $\Omega_p^{(g,m,n)}(V_1,\cdots V_m, W_1,\cdots W_n)$

① As in bosonic string theory, introduce $(m+n)$ disks D_a with coord. w_a, $2g-2+m+n$ spheres S_i with coord. z_i, $3g-3+2(m+n)$ circles C_s.

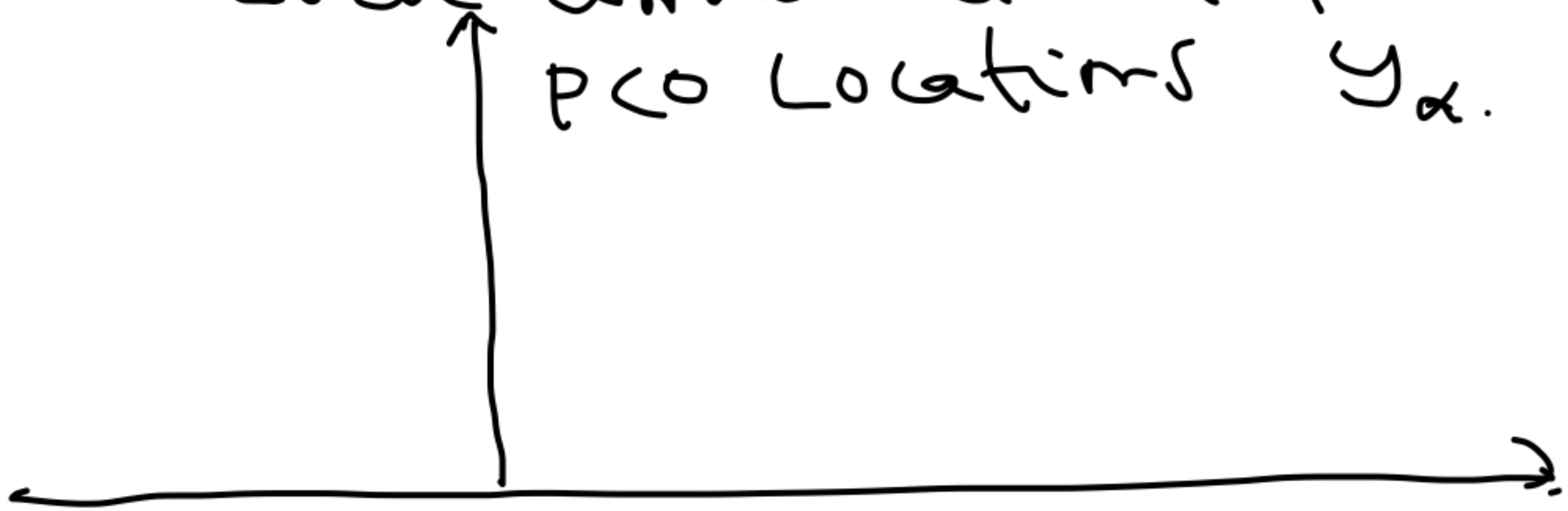
Transition fn. $\sigma_s = F_s(\tau_s)$

② Extra ingredient: Locations y_α of $2g-2+m+\frac{n}{2}$ PCO's. (in w_a coordinate if on D_a, in z_i coord. if on S_i)

③ Introduce the fiber bundle $\mathcal{P}_{g,m,n}$.

Local coordinate at punctures

pco locations y_α.



$\mathcal{M}_{g,m,n}$

$\{t^m\}$: coordinates on $\mathcal{P}_{g,m,n}$

$$\sigma_s = F_s(\tau_s, \vec{t}\,), \quad y_\alpha(\vec{t}\,)$$

$$B_m = \sum_s \oint_{c_s} \frac{\partial F_s(\tau_s, \vec{t})}{\partial t^m} b(\sigma_s)\, d\sigma_s \quad \Big| \begin{array}{l} \beta, \gamma. \\ \to \xi, \eta, \phi. \end{array}$$

$$+ \sum_s \oint_{c_s} \frac{\overline{\partial F_s(\tau_s, \vec{t})}}{\partial t^m} \bar{b}(\bar{\sigma}_s)\, d\bar{\sigma}_s$$

$$- \sum_\alpha \frac{1}{x(y_\alpha)} \frac{\partial y_\alpha(\vec{t})}{\partial t^m} \partial \xi(y_\alpha) \Rightarrow \text{extra term.}$$

formal.

$$\Omega_p^{(g,m,n)}(V_1, \ldots V_m, W_1, \ldots W_n) = \sum_{m_1, \ldots m_p} dt^{m_1} \wedge \ldots dt^{m_p}$$

$$\Big\langle B_{m^1} \ldots B_{m^p} \prod_{\beta=1}^{2g-2+m+\frac{n}{2}} x(y_\beta)\, V_1 \ldots V_m\, W_1 \ldots W_n \Big\rangle_{\Sigma_{g,m,n}}$$

$\to$ well defined expression$\Big]$ $\times (-2\pi i)^{-(3g-3+m+n)}$

$\Omega_p^{(g,m,n)}$ satisfies

$$\Omega_p^{(g,m,n)}(Q_B V_1, \cdots W_m) + \cdots \cdots$$

$$= (-1)^p \, d\Omega_{p-1}^{(g,m,n)}(V_1, \cdots W_m)$$

$$A(V_1, \cdots V_m, W_1, \cdots W_n)$$

$$= (g_s)^{3g-3+m+n} \int \Omega_{6g-6+2(m+n)}^{(g,m,n)}(V_1, \cdots V_m, W_1, \cdots W_n)$$

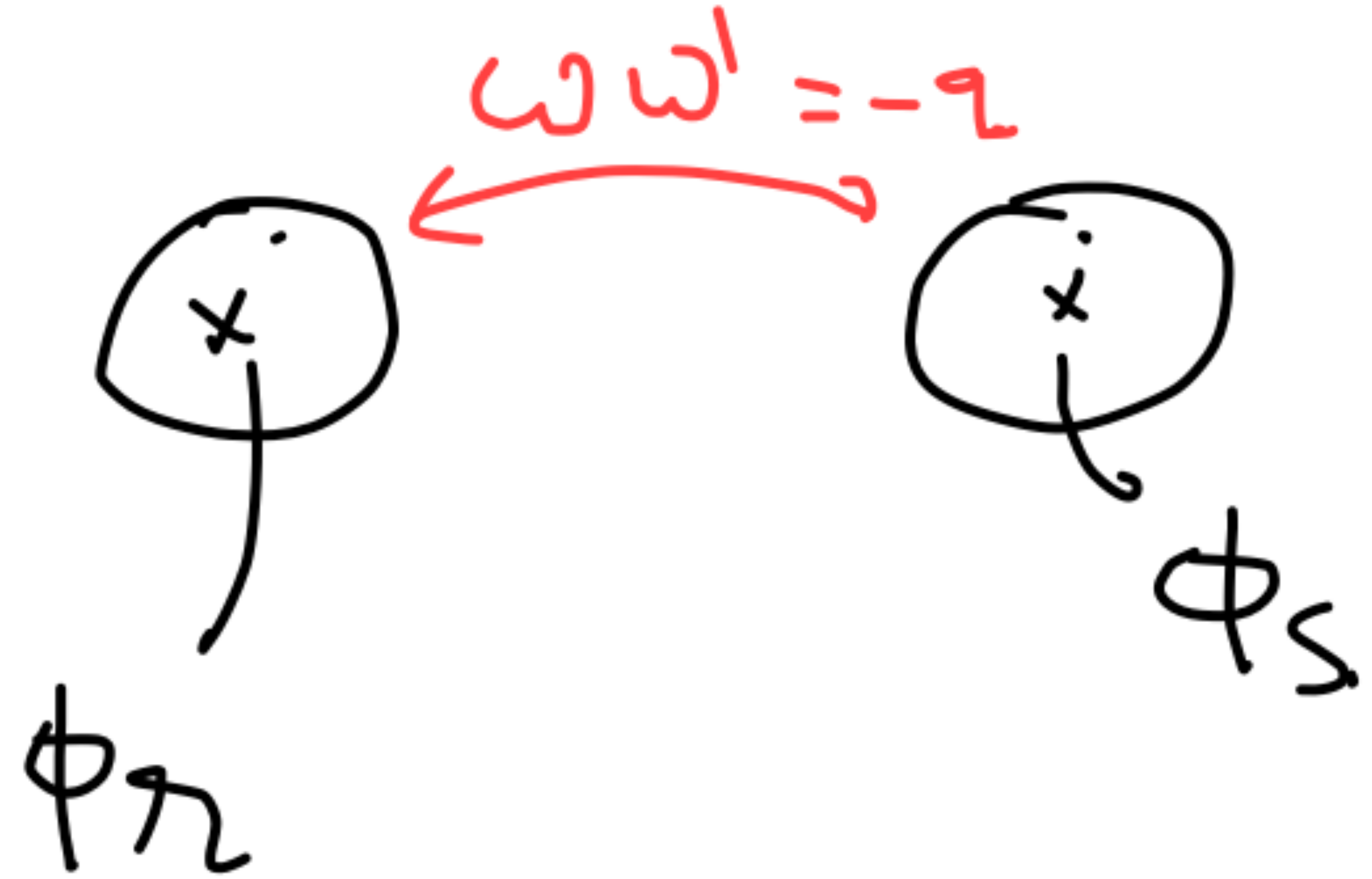
$S_{g,m,n}$

section $\swarrow$

$P_{g,m,n}$

Once we have this, we can now express the amplitude as sum over Feynman diagrams.

Procedure: Same as in bosonic string theory, but with one difference.



$$\Rightarrow \int dq \wedge d\bar{q} \; \langle \phi_r^c | b_0 \bar{b}_0 | \phi_s^c \rangle \, q^{h_s} \bar{q}^{\bar{h}_s}$$

For NS punctures, ϕ_r, ϕ_s have $p = -1$

$\Rightarrow \phi_r^c, \phi_s^c$ have picture no. -1 ✓

For R puncture: ϕ_r, ϕ_s have $p = -\frac{1}{2}$

$\Rightarrow \phi_r^c$ and ϕ_s^c have $p = -\frac{3}{2} \Rightarrow \langle \phi_r^c | b_0 \bar{b}_0 | \phi_s^c \rangle$ will vanish.

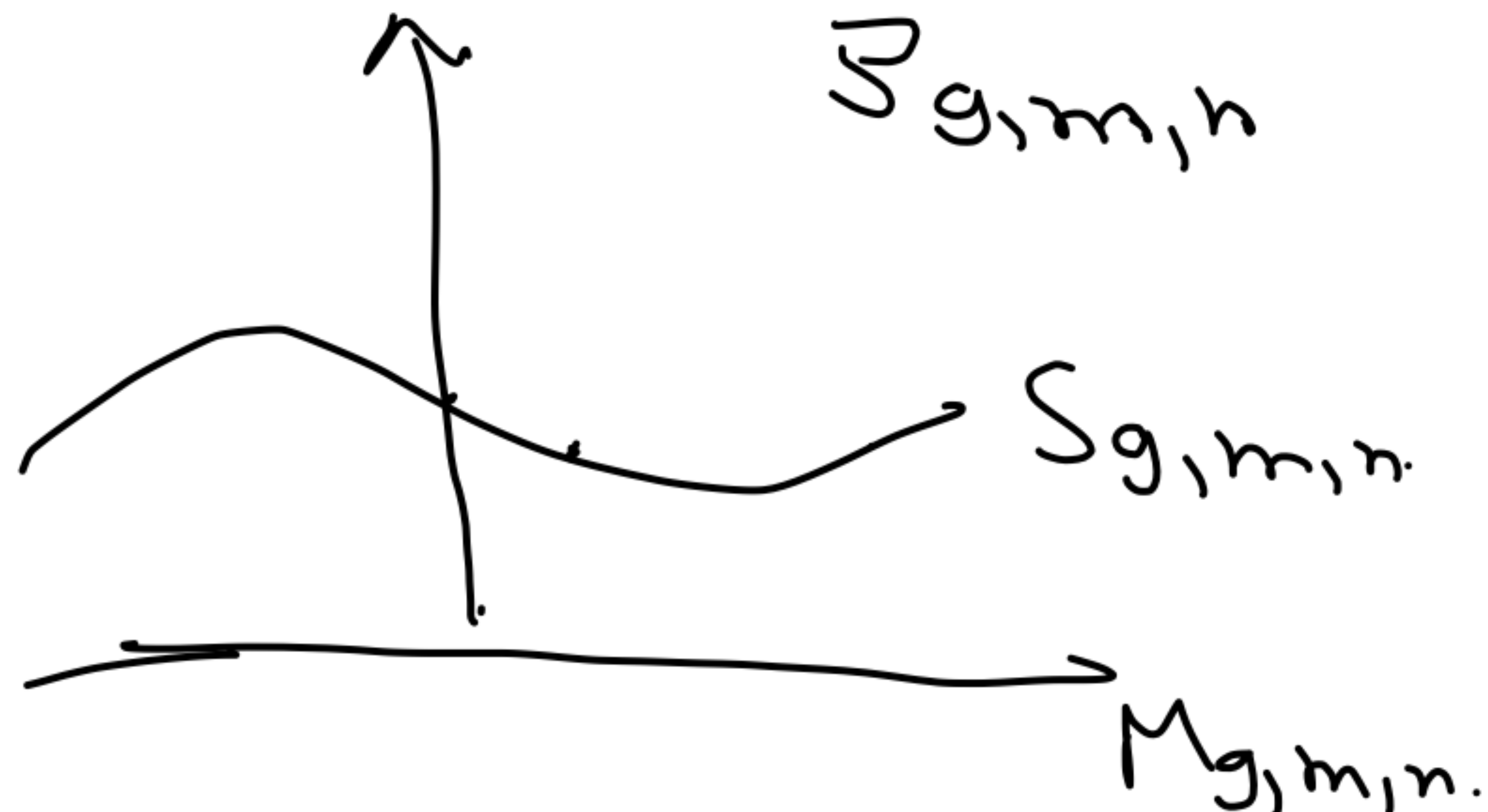
We avoid this by inserting one PCO inside $\langle \phi_r^c | \; | \phi_s^c \rangle$

We insert:

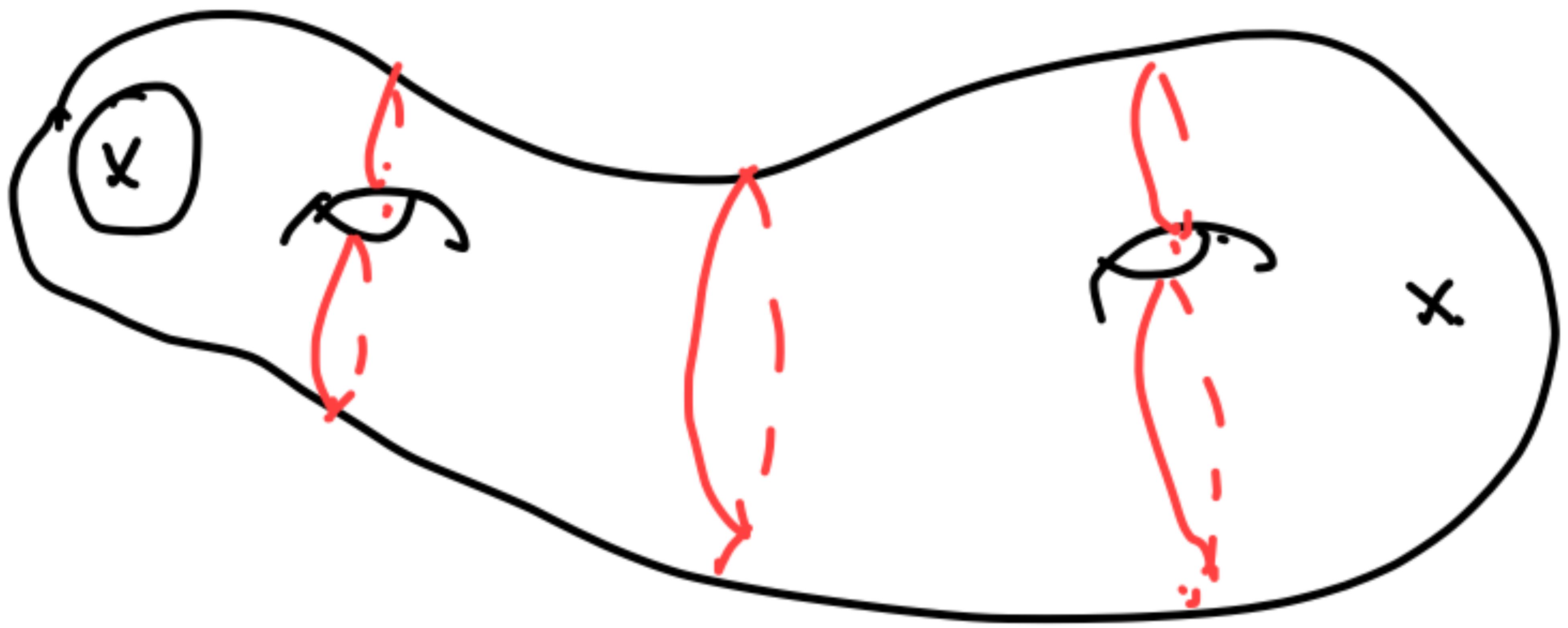
$$X_0 = \oint \frac{d\omega}{\omega} X(\omega)$$

→ Average of PCO insertions on a circle.

$S_{g,m,n}$: specifies local coords & PCO locations



One PCO insertion is average of all possible insertions on $|\omega| = |q|^{1/2}$ circle.



$(3g-3+m+n)$ circles C_{s} on $S_i \cap S_j$

$\sigma_s = F_s(\tau_s)$, $\boxed{\sigma_s = -\frac{q_s}{\tau_s}}$

$\sigma_s = \tau_s + a_s$ on $S_i \cap D_a$

$(3g-3+m+n)$ $q_s \Rightarrow 6g-6+2(m+n)$ real parameters.