Summary

① Take a pair of punctures with local coordinates w, w' (same or different surfaces)

② Sew them using $w w' = -q$

$\Rightarrow$ a new surface

For correlation fn. this is achieved by inserting:

$$\phi_r(0)\ \phi_s(0)\ \langle \phi_r^c | \phi_s^c \rangle\ q^{h_s}\ \bar{q}^{\bar{h}_s}$$

In w coordinate → in w' coordinate.

Repeated application $\Rightarrow$ correlators on any Riemann surface.

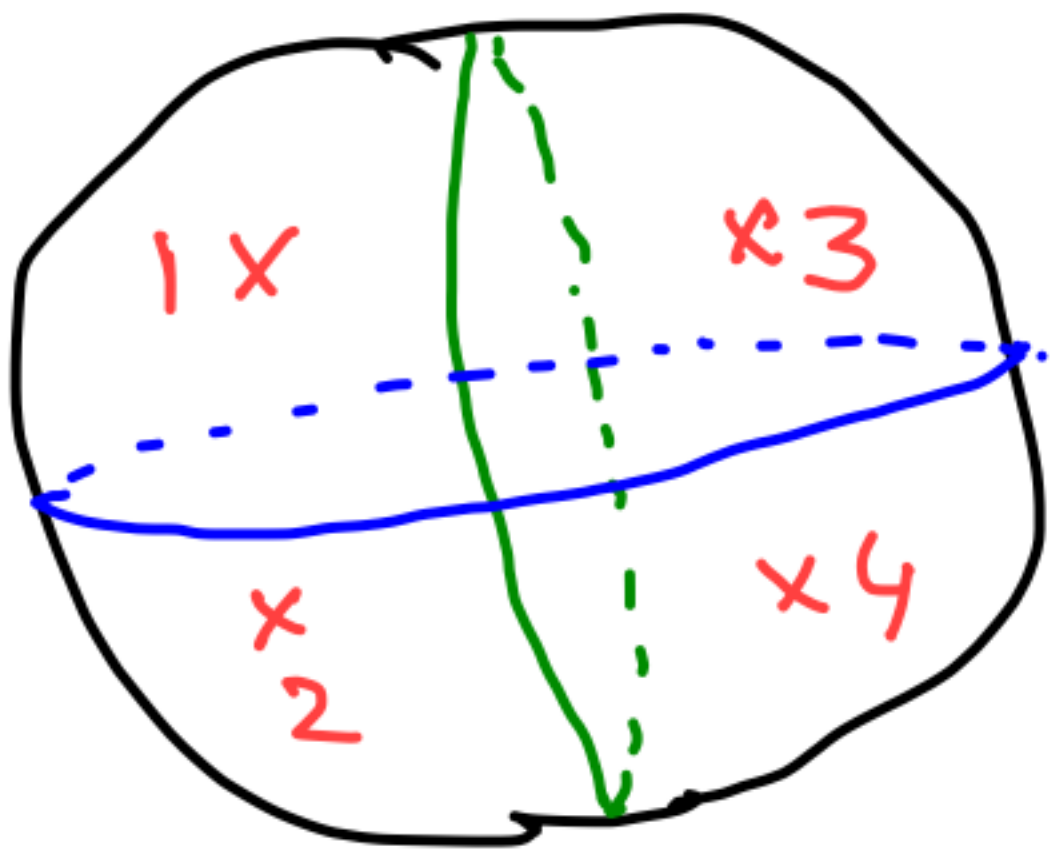
In practice one can combine this with analyticity and OPE to find closed form expression for correlators.

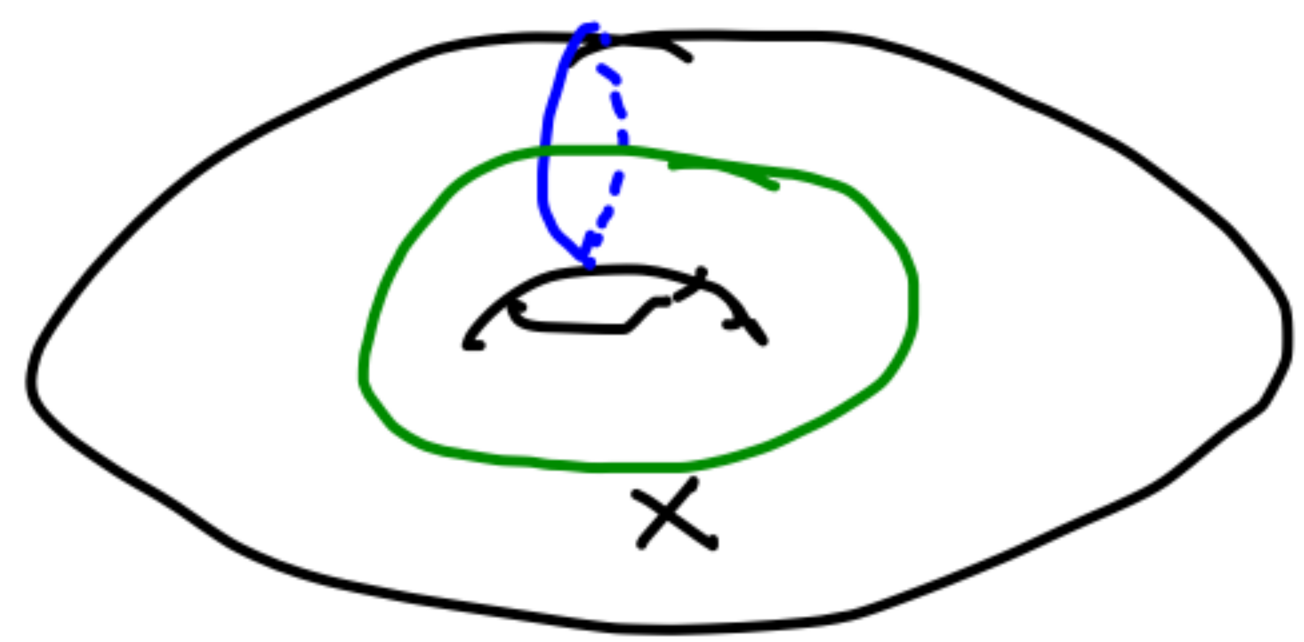
$\Rightarrow$ All correlators are obtained from knowledge of conformal weights and 3-point fn. on sphere.

Procedure of sewing $\rightarrow$ plumbing fixture

Consistency check

The same Riemann surface with punctures may be built from sphere 3-pt fn. in more than one way.





This is part of the consistency requirement of CFT

In principle there are ∞ no. of such consistency checks.

$\Rightarrow$ # constraints is infinite.

Result: As long sphere 4 pt. fn. and torus one point fn. are consistent, all higher genus correlation fn.s are also consistent.

Ex. Prove that on a genus g Riemann surface we need total ghost no. $-(6g-6)$ to get a non-zero correlator.

Hint: Use ghost no. of ϕ_r^c

$= 6 -$ ghost no. of ϕ_r

$$\langle \phi_r^c | \phi_s \rangle = \delta_{rs}$$

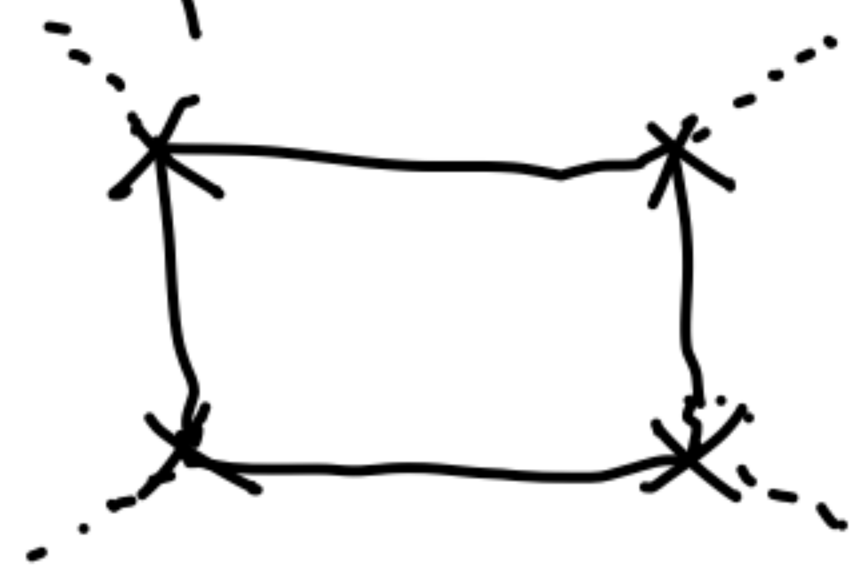
whiteboardApp.org

Q. Why are we interested in correlation fn. of higher genus surfaces?

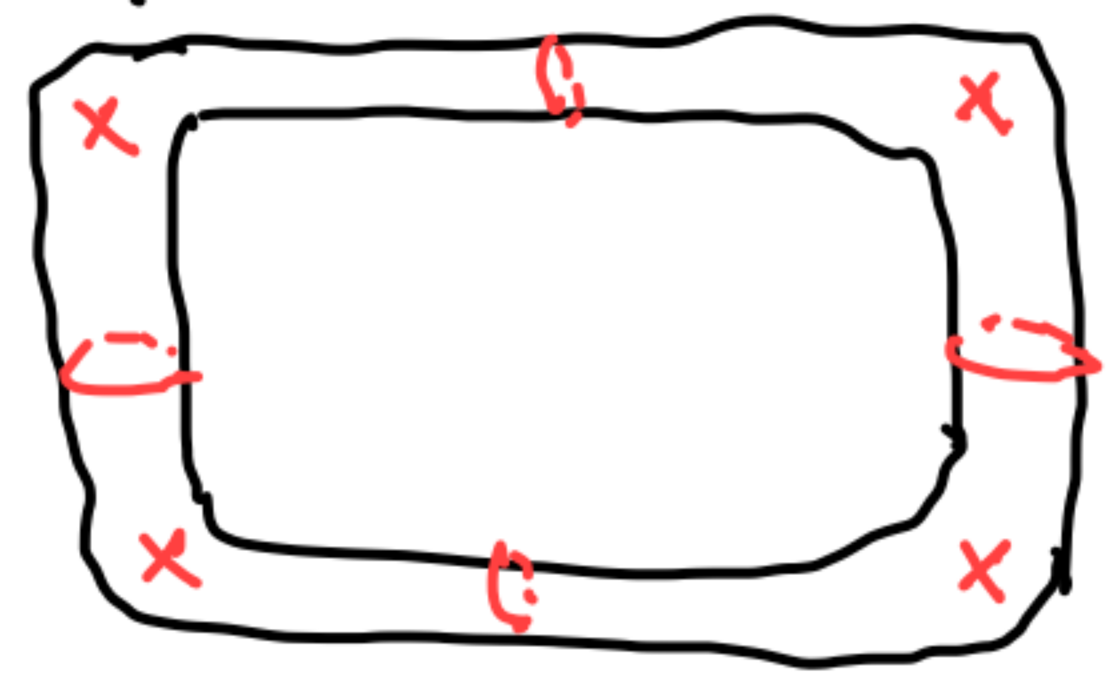
A. A g-loop amplitude in string theory is expressed in terms of CFT correlators on genus g surface.

<u>Intuitive understanding</u>

Example I: 1 loop, 4 pt. amplitude in QFT

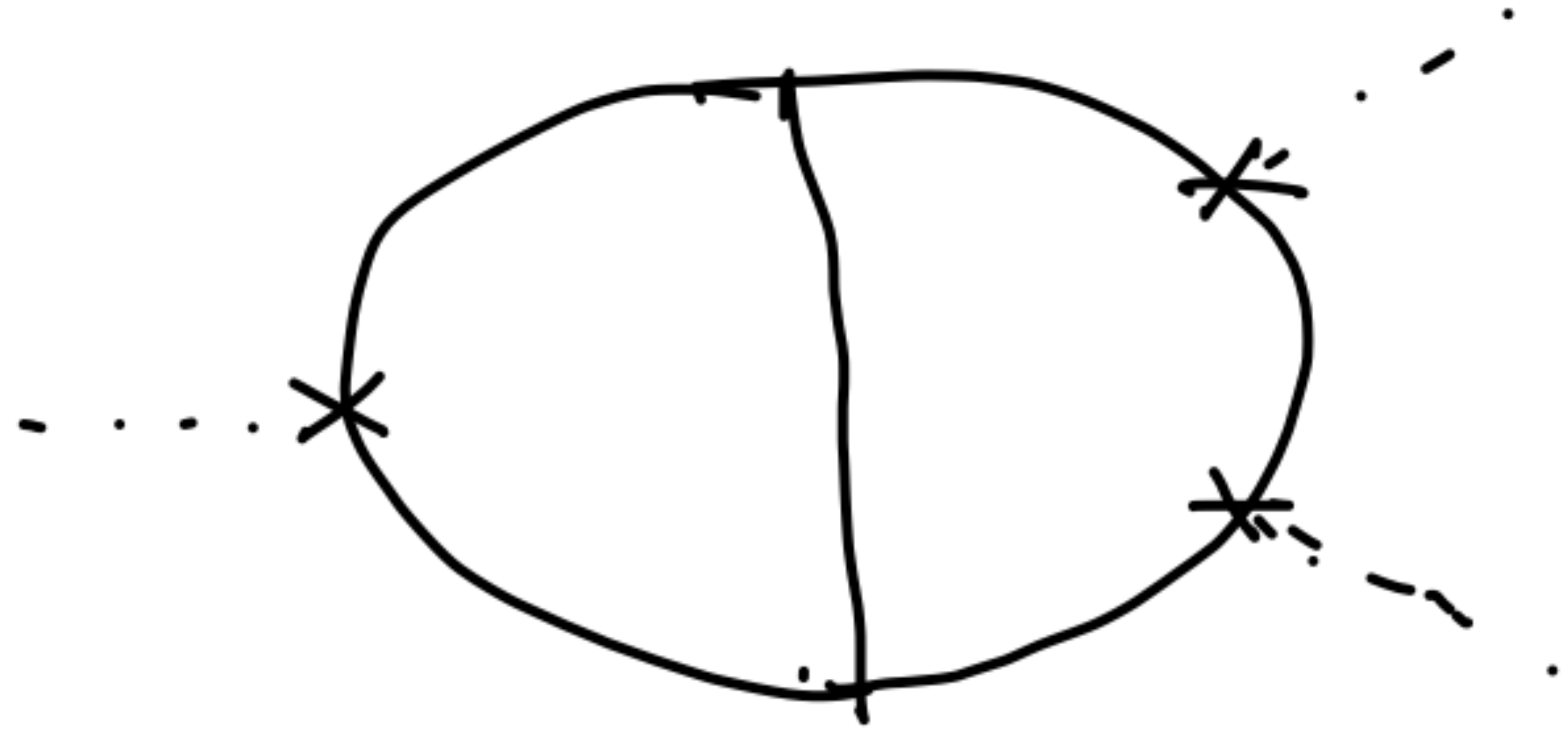


fatten $\longrightarrow$ internal lines.

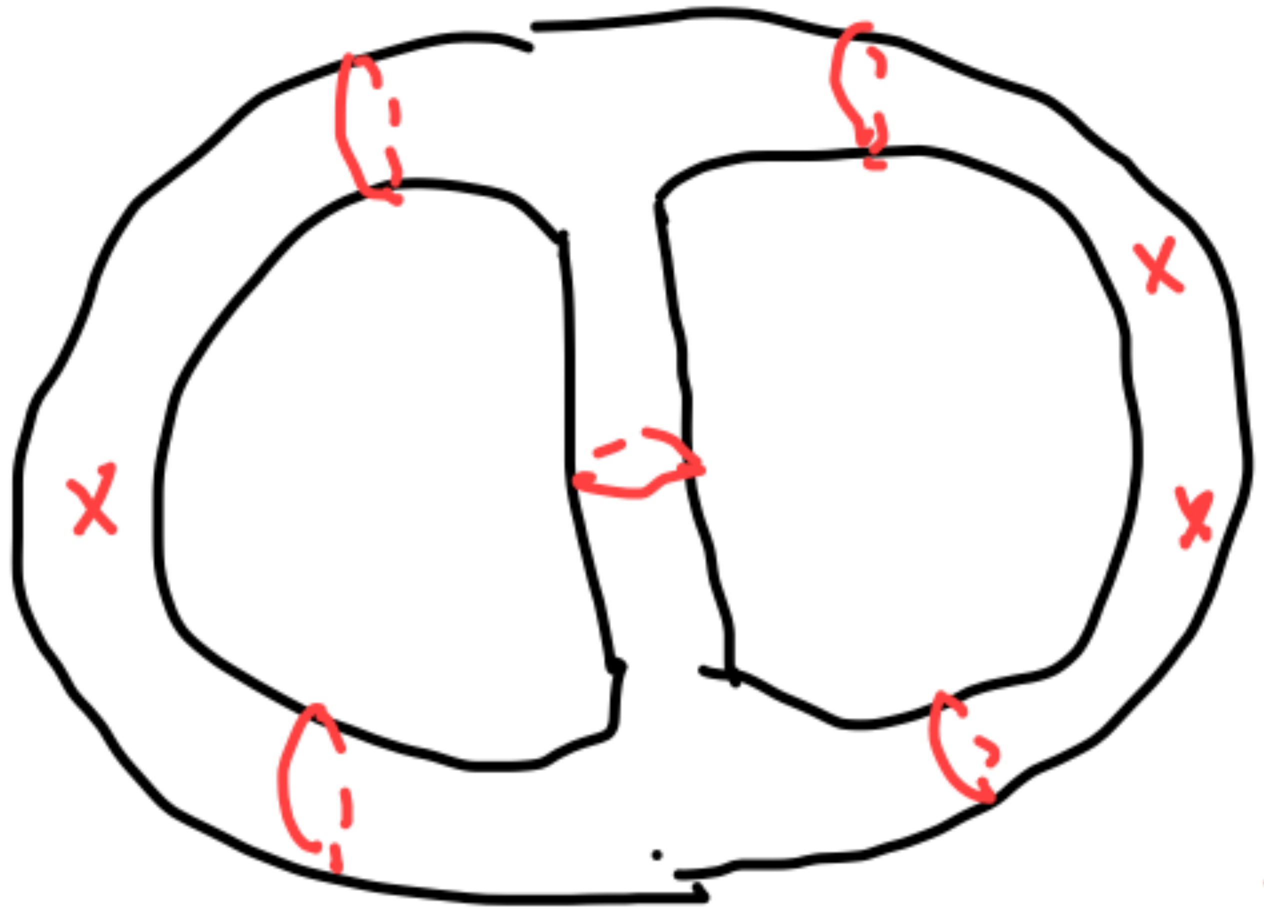


$\Rightarrow$ 4-pt fn. on torus.

Example 2. Two loop, 3-point fn



$\Downarrow$ Fatten



$\Rightarrow$ Genus 2, 3 point fn

What is the precise dictionary?

(a) What is computed? (QFT language)

(b) How is it computed using CFT correlators?

$\rightarrow$ Off-shell Green's fn. with external tree level propagators removed.

will be called Amplitude

$\Rightarrow$ Green's fn.

$\Downarrow$ LSZ

S-matrix

Defn. of off-shell string state:

— A state $|V\rangle = V(0)|0\rangle$ in the CFT

satisfying:

$b_0^-|V\rangle = 0, \quad L_0^-|V\rangle = 0.$ not necessarily $Q_B|V\rangle = 0$

$\Downarrow$ on-shell condition

$\Downarrow$

How to compute

$A(V_1, \cdots, V_n)$?

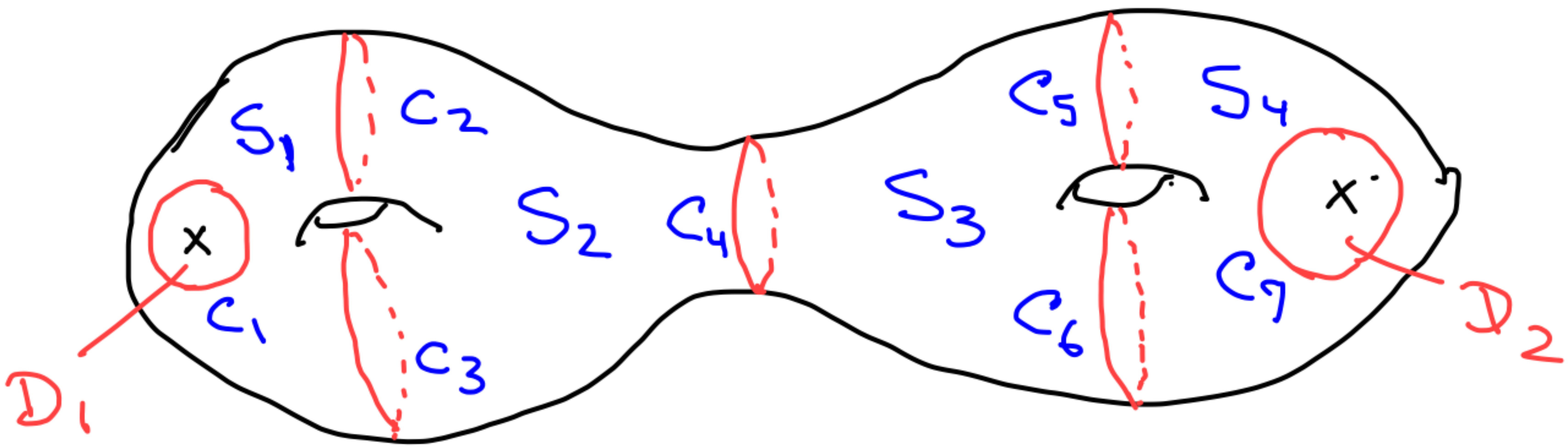
Begin with a general description of Riemann surface of genus g with n-punctures:

→ can be regarded as a union of n disks, one around each puncture, and $2g-2+n$ spheres, each with 3 holes, joined along $3g-3+2n$ circles.

Disks: D_a, $a = 1, \ldots n$, spheres: S_i $i = 1, \ldots 2g-2+n$

Circles: C_s : $s = 1, \ldots 3g-3+2n$. $\to S_i \cap S_j$ or $S_i \cap D_a$

Example: $g = 2, \quad n = 2.$



# of spheres $2g-2+n = 4$

# of circles $3g-3+2n = 7$

Fix some orientation on each circle C_s (arbitrary)

Introduce complex coordinate system:

w_a on D_a with $w_a = 0$ being the puncture

z_i on S_i

σ_s: coordinate on the left of C_s (z_i or w_a)

τ_s: coordinate on the right of C_s (z_i or w_a)

$\tau_s = F_s(\sigma_s) \rightarrow$ specifies the Riemann surface.

$\{F_s(\tau_s)\}$ is considered equivalent to $\{\tilde{F}_s(\tau_s)\}$ if they can be related by coord. tr. of the form:

$$z_i \to h_i(z_i), \quad w_a \to \tilde{h}_a(w_a), \quad \tilde{h}_a(0) \equiv 0$$

space of equivalence classes is finite dimensional & known as the moduli space of genus g Riemann surface with n punctures.

$\Downarrow$

$6g-6+2n$

$\mathcal{M}_{g,n}$

Introduce another space $\mathcal{P}_{g,n}$

Equivalence classes of $\{F_s(\tau_s)\}$

under coordinate trs. $z_i \to h_i(z_i)$

and $w_a \to e^{i\alpha_a} w_a$, α_a are

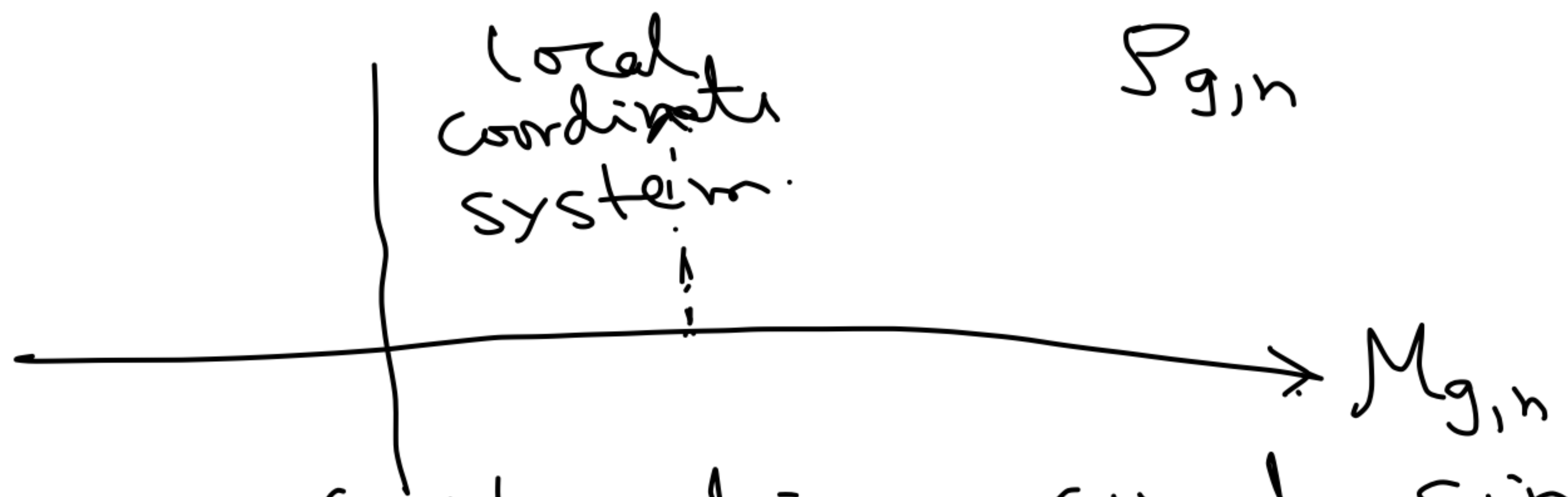
arbitrary constants.

Physically $\mathcal{P}_{g,n}$ contains information

about $\mathcal{M}_{g,n}$ and choice of local

coordinate at the puncture.

$\mathcal{P}_{g,n}$ can be regarded as a fiber bundle with base $M_{g,n}$ and fiber containing choice of local coordinates.



$\mathcal{P}_{g,n}$ is infinite dimensional since it has info. about n independent f-s
$\to$ choice of n local coordinates up to phase.

$\{t^m\} \equiv \vec{t}$: coordinates on $\mathcal{P}_{g,n}$

∞ # of coordinates

Given $\vec{t}$, we have a given set of fr.s $F_s(\tau_s;\vec{t})$

not unique since it can be changed by $z_i \to h_i(z_i)$ or $w_a \to e^{i\alpha_a} w_a$.

Pick some representative F_s.

$$\sigma_s = F_s(\tau_s;\vec{t})$$

Define a CFT "operator"

$$B_m = \sum_{s=1}^{3g-3+2n} \left[\oint_{C_s} \frac{\partial F_s(\tau_s, \vec{t})}{\partial t^m} \, d\sigma_s \, b(\sigma_s) + \oint_{C_s} \frac{\partial \overline{F_s(\tau_s, \vec{t})}}{\partial t^m} \, d\bar{\sigma}_s \, \bar{b}(\bar{\sigma}_s) \right]$$

For a given set of off-shell string states $|V_1\rangle, \ldots, |V_n\rangle$ define a p form $\Omega_p^{(g,n)}(V_1, \ldots V_n)$ on $\mathcal{P}_{g,n}$ as follows:

$$\Omega_p^{(g,n)}(V_1, \cdots V_n) = \sum_{m_1, \cdots, m_p} \langle B_{m_1} \cdots B_{m_p} V_1 \cdots V_n \rangle_{\Sigma_{g,n}} (-2\pi i)^{-(3g-3+n)} dt^{m_1} \wedge \cdots \wedge dt^{m_p}$$

$w_1 = 0$ $\quad$ $w_n = 0$

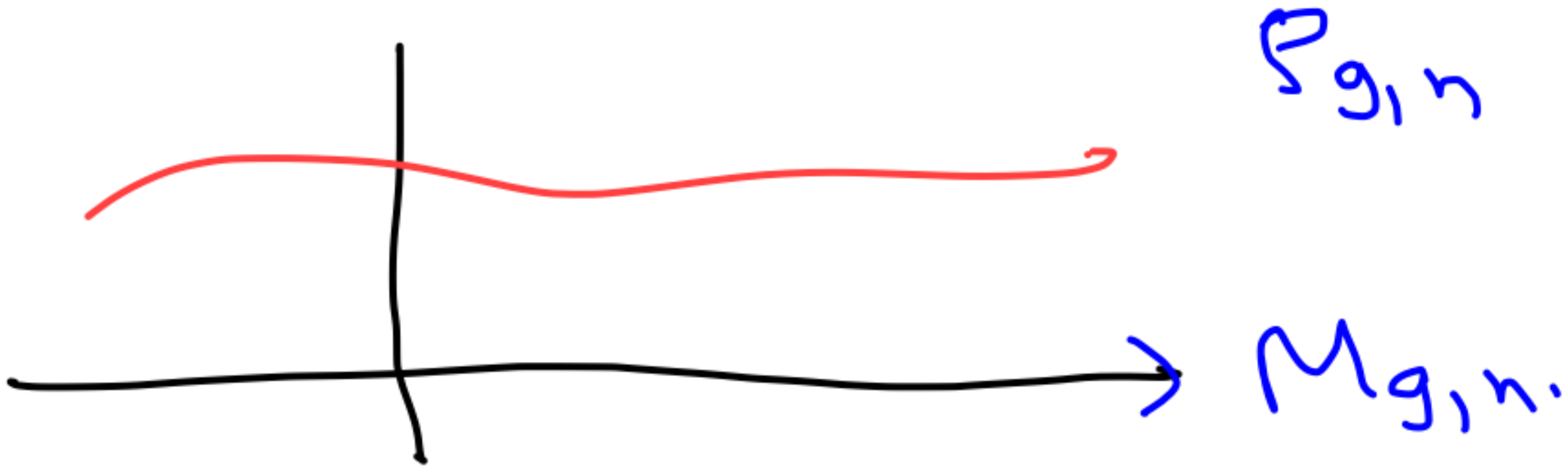
$\Sigma_{g,n}$: Genus g Riemann surface with n-puncture and given choice of local coords. at punctures determined by the point at $\mathcal{P}_{g,n}$.

① This description is manifestly invariant under change of coordinate in $\mathcal{P}_{g,n}$ $t^i \to f^i(\vec{t})$

② If we want to pull back $\Omega_p^{(g,n)}$ on some p-dimensional subspace of $\mathcal{P}_{g,n}$ with intrinsic coordinates $u^1, \ldots u^p$, then the result is obtained simply by replacing t^m by u^i's.

Defn. of $A(V_1, \dots V_n)$.

① choose a section $S_{g,n}$ of $\mathcal{S}_{g,n}$.



② $$A(V_1, \dots, V_n) = (g_s)^{2g-2+n} \int_{S_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1, \dots V_n).$$

Tree level S-matrix: iA

Loop level S-matrix: Requires. LSZ

We'll not derive this formula but we'll check various consistency conditions.

Crucial identity (derived using CFT properties)

$$\Omega_p^{(g,n)}(Q_B V_1, V_2, \cdots V_n) + (-1)^{V_1}\Omega_p^{(g,n)}(V_1, Q_B V_2, \cdots V_n) + \cdots + \Omega_p^{(g,n)}(V_1, \cdots V_{n-1}, Q_B V_n)(-1)^{V_1}(-1)^{V_2}\cdots(-1)^{V_{n-1}}$$

$$= (-1)^p \, d\Omega_{p-1}^{(g,n)}(V_1, \cdots, V_n)$$

$$Q_B V(z,\bar z) = \oint_z j_B(w) V(z,\bar z) + \oint_{\bar z} \bar j_B(\bar w) V(z,\bar z)$$

$$Q_B V(0)|0\rangle = Q_B |V\rangle$$

w $\times z$

Ex: $\{Q_B, b(z)\} = T(z)$, $\{Q_B, \bar{b}(\bar{z})\} = \bar{T}(\bar{z})$

$\oint \epsilon(z) T(z)\, dz$ generates $z \to z + \epsilon(z)$

Normalization

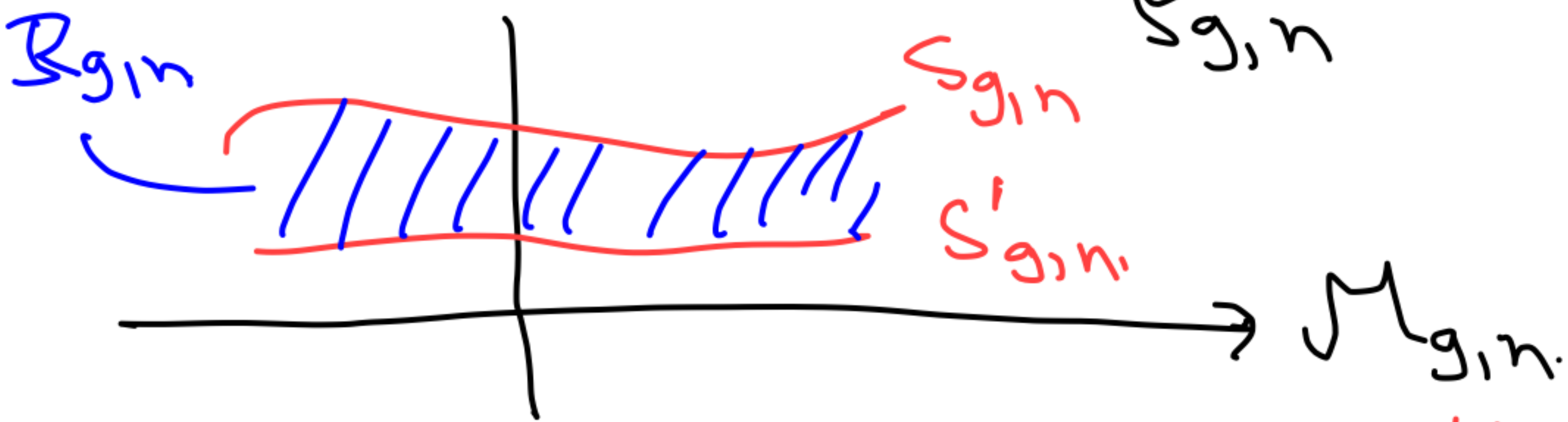
$\oint$ is defined such that $\oint \frac{dz}{z} = 1$

$\oint \frac{d\bar{z}}{\bar{z}} = 1$.

includes $\frac{1}{2\pi i}$ factor.

Consequences

(1) Dependence on the section $S_{g,n}$



This has to be supplemented by separate analysis of contribution from $\partial M_{g,n}$.

$$\int_{S_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1,\dots,V_n) - \int_{S'_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1,\dots,V_n)$$

$$= \int_{R_{g,n}} d\Omega^{(g,n)}_{6g-6+2n}(V_1,\dots,V_n) = -\int_{R_{g,n}} \Big[\Omega^{(g,n)}_{6g-6+2n+1}(Q_B V_1, V_2,\dots,V_n) + (-1)^{V_1}\Omega^{(g,n)}_{6g-6+2n+1}(V_1, Q_B V_2, V_3,\dots,V_n) + \dots\Big]$$

$= 0$ if $Q_B V_i = 0$ $\forall i$

② $A(Q_B\Lambda, V_2, V_3, \cdots V_n)$ $\quad Q_B V_i = 0$ for $i=2,\cdots,n$

$$= \int_{S_{g,n}} \Big[\Omega^{(g,n)}_{6g-6+2n}(Q_B\Lambda, V_2, V_3, \cdots V_n)$$

$$+ \Omega^{(g,n)}_{6g-6+2n}(\Lambda, Q_B V_2, \cdots V_n)(-1)^{\Lambda}$$

= 0 anyway.

$$+ \cdots \Big]$$

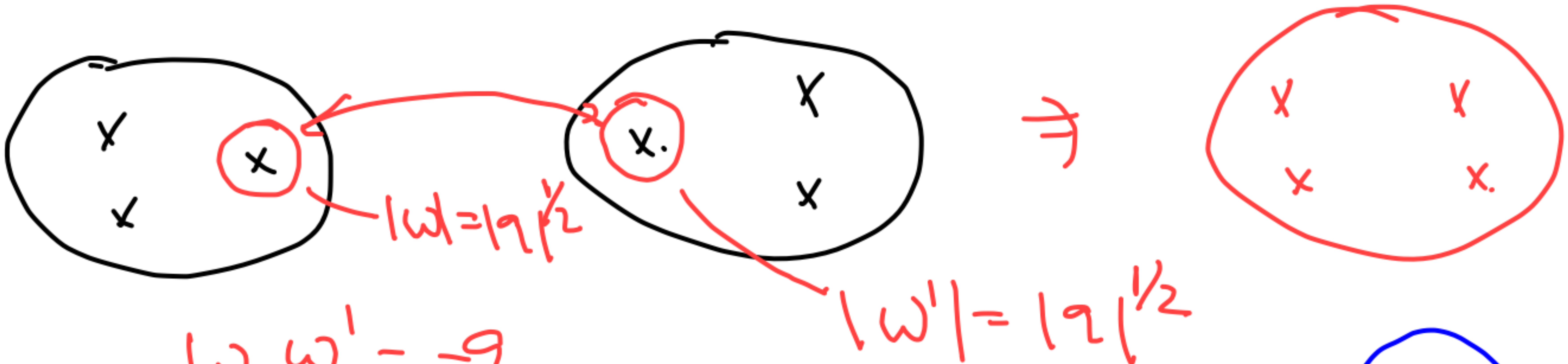
$$= \int_{S_{g,n}} d\Omega^{(g,n)}_{6g-6+2n-1}(\Lambda, V_2, \cdots V_n)$$

$= 0$ up to contributions from $\partial \mathcal{M}_{g,n}$

to be analyzed separately.

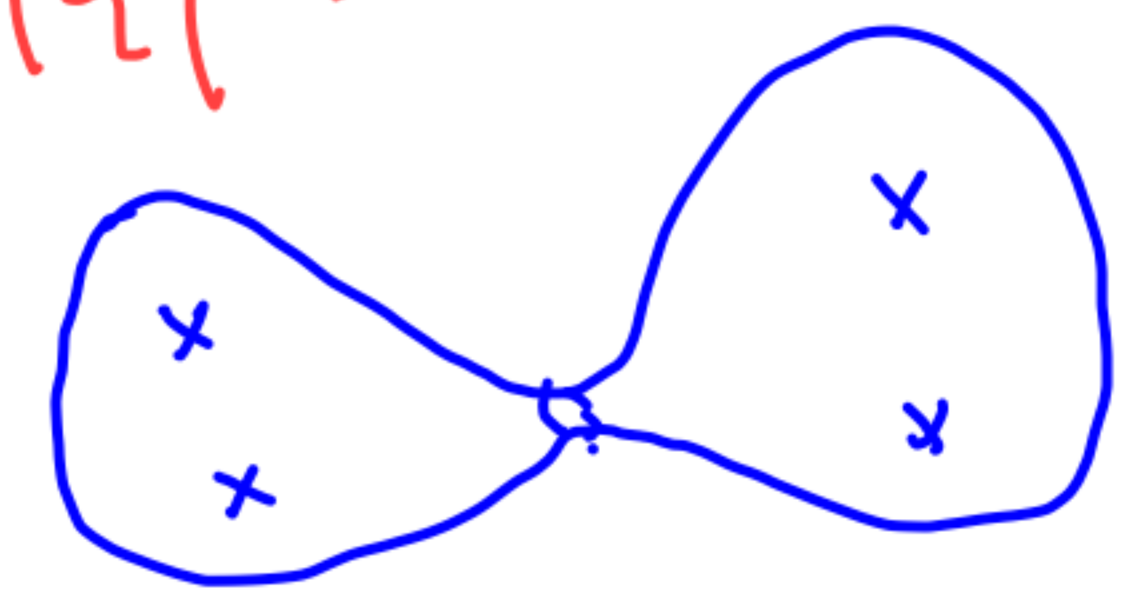
Pure gauge states decouple.

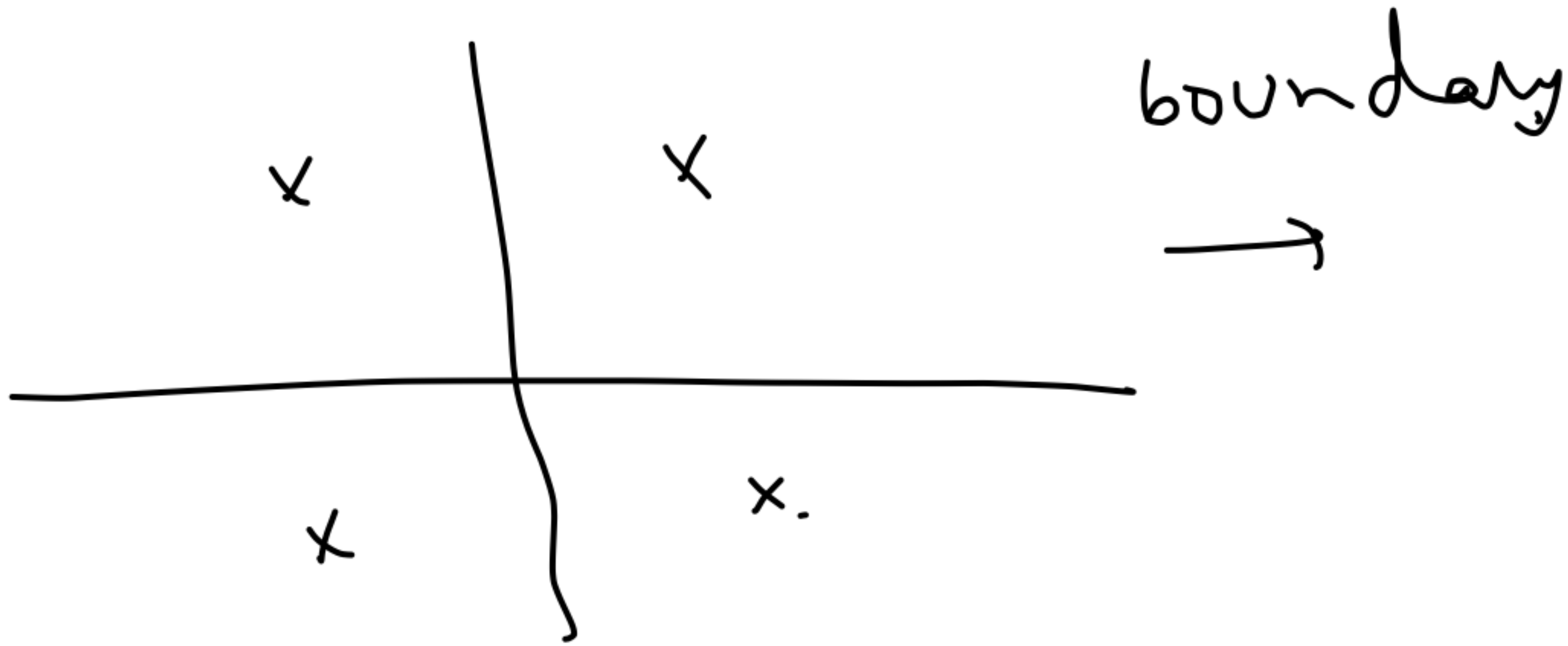
⇒ Proof of general coordinate invariance and other gauge invariances in the interacting theory.



$w w' = -q$

$q \to 0$ limit ⇒ boundary of $M_{0,4}$



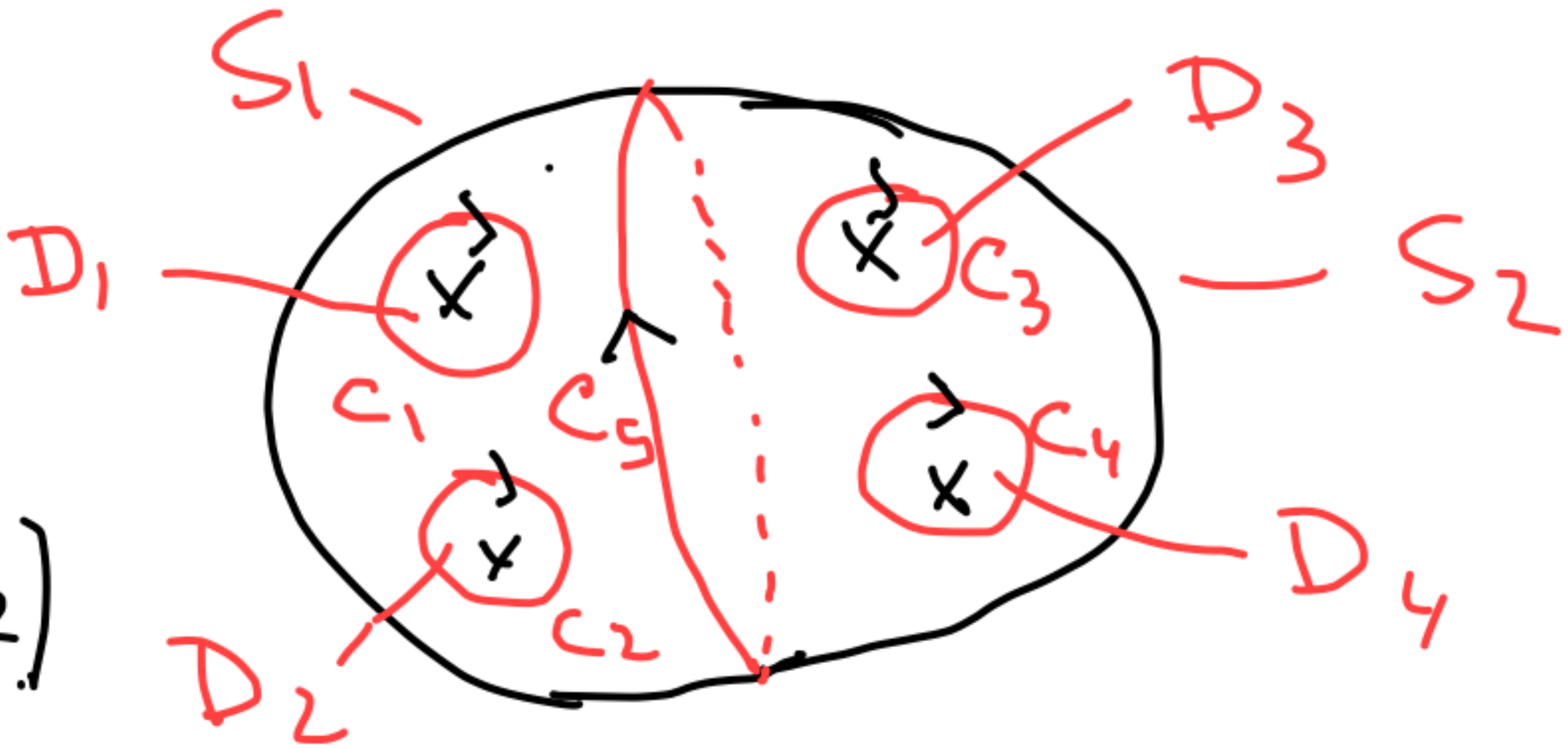


$\left.\begin{matrix} x \\ x \end{matrix}\right] \Rightarrow q \to 0$ limit of the previous diagrams

We'll apply this to sphere
four point fn. $\mathcal{M}_{0,4}$

$\rightarrow$ $6g-6+2n = 2$ dim. moduli space.

On-shell amplitudes



C_5:
$z_1 = F_5(z_2)$
$= -q/z_2$

On C_1:
$z_1 = F_1(\omega_1)$

On C_2
$z_1 = F_2(\omega_2)$

C_3: $z_2 = F_3(\omega_3)$

C_4: $z_2 = F_4(\omega_4)$