Superstring Perturbation Theory

1. https: home.icts.res.in/~sen/

   course on Superstring Perturbation theory

2. arXiv: 1703.06410 (sections 2, 3)

String theory: Elementary constituents of matter are one dimensional objects (strings)

- Radical idea, but why?

① String theory contains gravity and possibly the standard model of particle phy...

② Unlike QFTs, string theory is free fr... ultraviolet divergences.

$\rightsquigarrow$ UV finite, unambiguous theory of quantum gravity.

Goal: To give a glimpse of how string theory works. (skip many derivations) Exercises can be done based on the results that we provide.

Starting point: Action of a relativistic point particle.

$$S = -m\int ds = -m\int d\xi \sqrt{-\frac{dX^\mu}{d\xi}\frac{dX^\nu}{d\xi}\eta_{\mu\nu}}$$

$ds \to$ proper time

$\xi \to$ parameter along the world line

$(\xi^0, \xi^1) : \{\xi^\alpha\} \quad \alpha = 1, 2.$

Action (Nambu-Goto)

$$S = -T \int d^2\xi \sqrt{-\det h}, \quad h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha} \qquad \alpha = 1, 2$$

"Area" of the world-sheet

$$T : \text{string tension} = \frac{1}{2\pi\alpha'}$$

We'll set $\hbar = 1$, $c = 1$, $\alpha' = 1$

We simplify the action by introducing an auxiliary set of variables $\gamma_{\alpha\beta}$

world-sheet metric

$$S_P = -\frac{1}{4\pi}\int \sqrt{-\det\gamma}\;\gamma^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu}$$

Matrix Inverse of $\gamma_{\alpha\beta}$

Ex. $\gamma_{\alpha\beta}$ eq. of motion $\Rightarrow$ $\gamma_{\alpha\beta} \propto \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$

substitute into $S_P \Rightarrow S_{NG}$

classically the two actions are equivalent

We quantize $S_P \rightarrow$ 2-D general coord. inv.

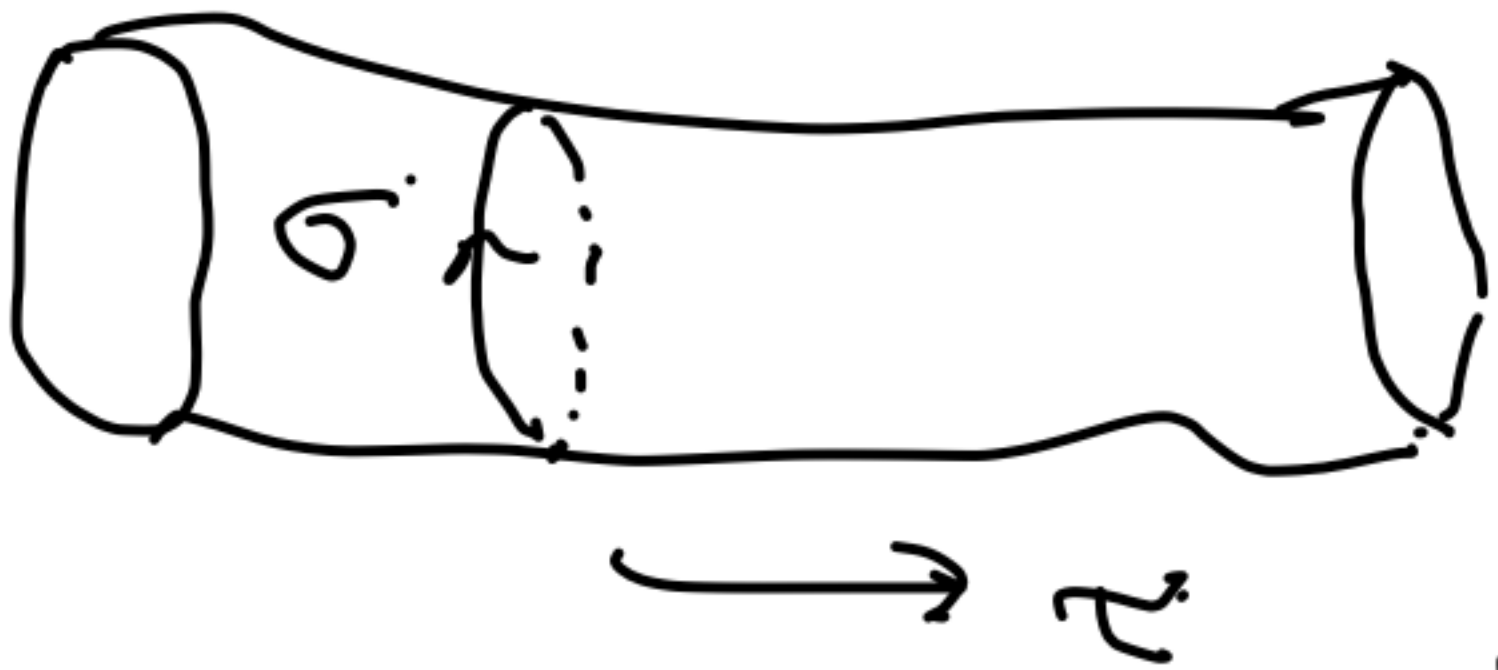
$\Rightarrow$ Gauge fix.

Choose gauge $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$.

Wick rotate the world-sheet and set

$\gamma_{\alpha\beta} = \delta_{\alpha\beta}$. | $\xi^0 = i\tau$ $\quad \xi^1 : \sigma \quad \sigma \equiv \sigma + 2\pi$

$\to$ Euclidean world-sheet time.



Complex coordinate

$w = \tau + i\sigma$

$$z = e^{w} = e^{\tau + i\sigma}$$

$\tau \to -\infty \Rightarrow z = 0$, $\tau \to \infty \Rightarrow z = \infty$

Maps the cylinder to a plane.

Add $z = \infty$ point to the complex plane $\to$ sphere

τ ordering $\leftrightarrow$ $|z|$ ordering (radial ordering)

Gauge fixed action

$$S = -\frac{1}{2\pi}\int d^2w \left[\partial X^\mu \bar\partial X^\nu \eta_{\mu\nu} + b\bar\partial c + \bar c \partial \bar b\right]$$

$\partial \equiv \frac{\partial}{\partial w}$, $\bar\partial = \frac{\partial}{\partial \bar w}$, $b, c, \bar b, \bar c$: ghosts.
(anti-commuting)

Eq. of motion:

$\partial\bar\partial X^\mu = 0$, $\bar\partial c = 0$, $\bar\partial b = 0$, $\partial \bar c = 0$, $\partial \bar b = 0$

$\Rightarrow$ ∂X^μ, b, c : holomorphic.
$\bar\partial X^\mu, \bar b, \bar c$: anti-holomorphic.

Mode expansions

$$X^\mu = -\frac{i}{\sqrt{2}}\left[\sum_{n\neq 0} \alpha^\mu_n e^{-nw} + \sum_{n\neq 0} \bar{\alpha}^\mu_n e^{-n\bar{w}}\right] + x^\mu + ip^\mu \tau$$

$$b = \sum_n b_n e^{-nw}, \quad c = \sum_n c_n e^{-nw}$$

$$\bar{b} = \sum_n \bar{b}_n e^{-n\bar{w}}, \quad \bar{c} = \sum_n \bar{c}_n e^{-n\bar{w}}$$

Equal time (anti-) commutator.

Ex. $[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$, $\{b_m, c_n\} = \delta_{m+n,0}$.

similarly for the anti-holomorphic ones.

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}.$$

Response of the theory to deformations of $\gamma_{\alpha\beta}$:

$\rightarrow$ captured in the energy-momentum tensor $T_{\alpha\beta}$.

Result: $T_{\omega\omega} \equiv T = -\eta_{\mu\nu}\partial X^\mu \partial X^\nu - 2b\partial c + c\partial b$

$T_{\bar{\omega}\bar{\omega}} \equiv \bar{T} = -\eta_{\mu\nu}\bar{\partial} X^\mu \bar{\partial} X^\nu - 2\bar{b}\bar{\partial}\bar{c} + \bar{c}\bar{\partial}\bar{b}$

$T_{\omega\bar{\omega}} = 0.$ $\Rightarrow$ Traceless $\Rightarrow$ Conformally inv. theory.

Conservation law $\partial^\alpha T_{\alpha\beta} = 0.$

$\partial_{\bar{\omega}} \underbrace{T_{\omega\omega}}_{\rightarrow \text{hol.}} + \partial_\omega \underbrace{T_{\bar{\omega}\omega}}_{=0} = 0$, $\quad \partial_\omega T_{\bar{\omega}\bar{\omega}} + \partial_{\bar{\omega}} T_{\omega\bar{\omega}} = 0.$

Suppose that $K(w)$ is a holomorphic field.

$$\partial_{\bar{w}} \left(f(w)\, K(w) \right) = 0$$

$\hookrightarrow$ any holomorphic fn.

$$\partial_{\bar{w}} \underset{\underset{f(w)\,K(w)}{|||}}{J_w} + \partial_w \underbrace{J_{\bar{w}}}_{\equiv 0} = 0 \quad \Rightarrow \quad Q_\tau = \oint_{\mathrm{Re}\, w = \tau} J_w \, dw$$

$$\propto \oint J^\tau d\sigma \rightarrow \text{Conserved since } \partial_\alpha J^\alpha = 0.$$

$$\Rightarrow Q_\tau = \oint_{\mathrm{Re}\, w = \tau} J_w \, dw \text{ is } \tau \text{ independent.}$$

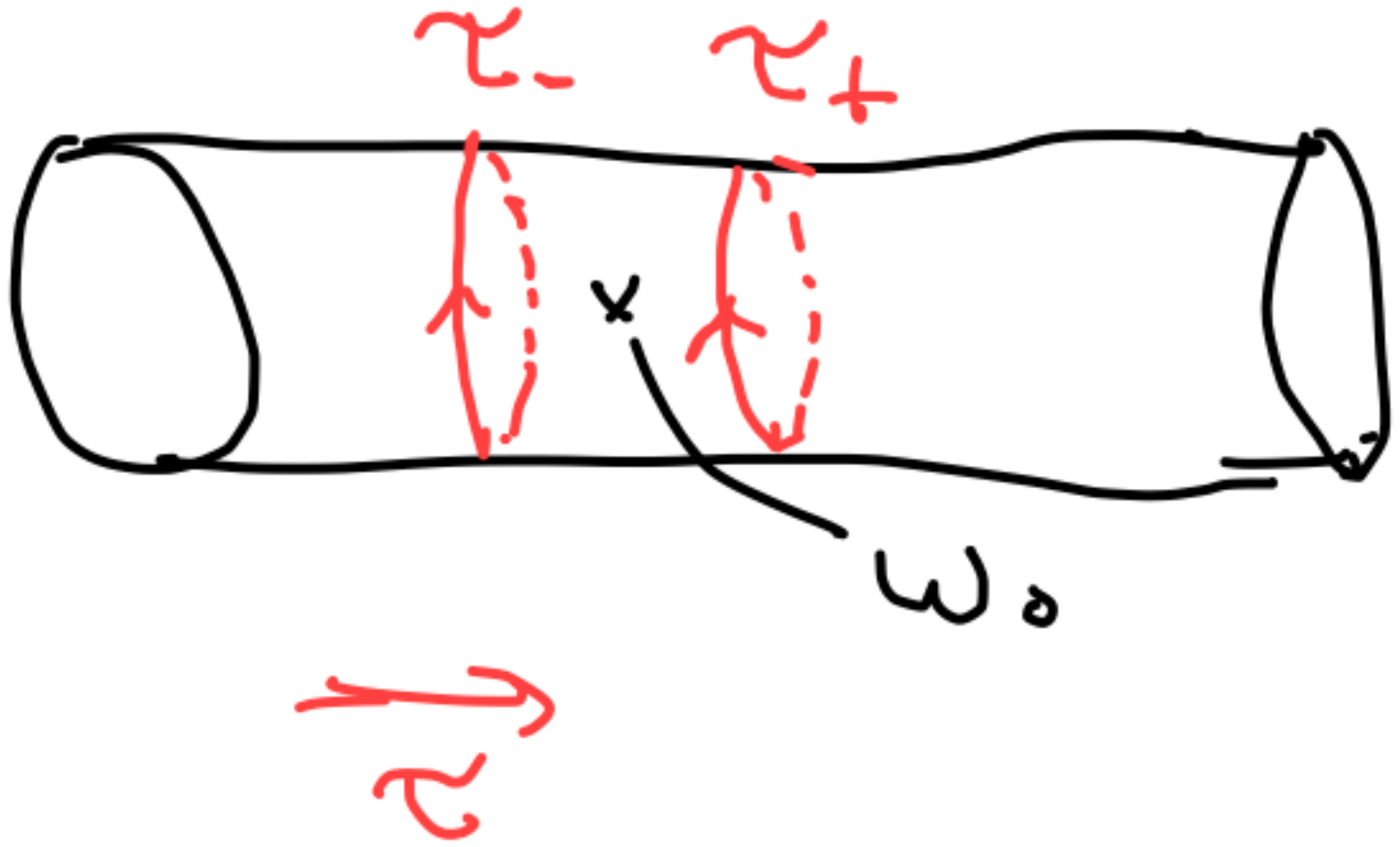
contains $\frac{1}{2\pi i}$ factor.

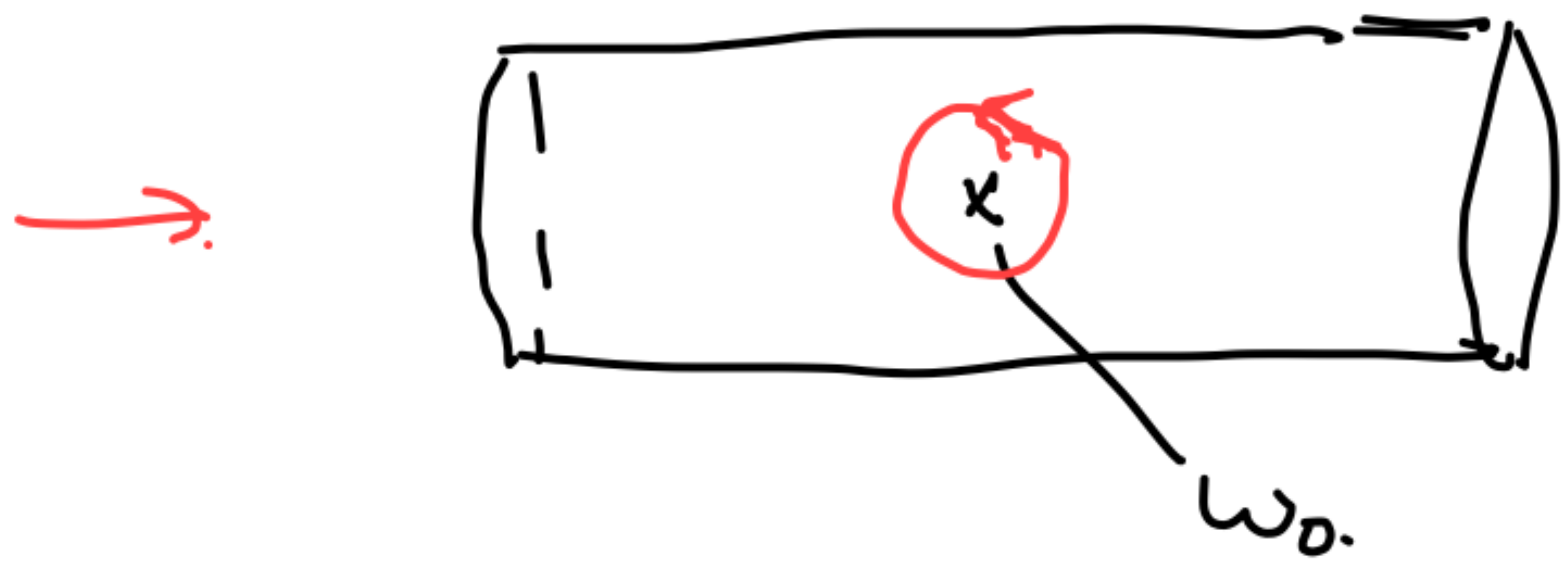
Suppose $\phi(w,\bar{w})$ is another operator.

$$[Q, \phi(w_0,\bar{w}_0)] = Q_{\tau_+}\, \phi(w_0,\bar{w}_0) - \phi(w_0,\bar{w}_0)\, Q_{\tau_-}$$

$\tau_+ > \mathrm{Re}\, w_0, \quad \tau_- < \mathrm{Re}\, w_0$

$$= \oint_{\mathrm{Re}\, w = \tau_+} T(J_w(w)\, \phi(w_0,\bar{w}_0)) - \oint_{\mathrm{Re}\, w = \tau_-} T(J_w(w)\, \phi(w_0,\bar{w}_0))$$





$$[Q, \phi(w_0, \bar{w}_0)] = \oint_{w_0} T(J_w(w)\, \phi(w_0, \bar{w}_0))$$

$\Rightarrow$ the coefficient of $\frac{1}{w - w_0}$ in the product $T(J_w(w)\, \phi(w_0, \bar{w}_0))$ gives $[Q, \phi(w_0, \bar{w}_0)]$.

Converse: Given a commutator we can determine singular terms in the operator product.

Ex. $b(\omega_1)\, c(\omega_2) \simeq \frac{1}{\omega_1-\omega_2}$, $\partial X^\mu(\omega_1)\, \partial X^\nu(\omega_2) \simeq -\frac{1}{2}\frac{\eta^{\mu\nu}}{(\omega_1-\omega_2)^2}$

$$\bar{b}(\bar{\omega}_1)\, \bar{c}(\bar{\omega}_2) \simeq \frac{1}{\bar{\omega}_1-\bar{\omega}_2}$$

$$\bar{\partial} X^\mu(\bar{\omega}_1)\, \bar{\partial} X^\nu(\bar{\omega}_2) \simeq -\frac{1}{2}\frac{\eta^{\mu\nu}}{(\bar{\omega}_1-\bar{\omega}_2)^2}$$

Convention: All products are inside time ordering.

Singularities in the operator product are the same in all coordinates (local properties)

$T_{w\bar{w}}=0 \Rightarrow$ Theory is conformally invariant

Conformal trs: $w \to f(w)$

locally holomorphic

A field $\phi(w,\bar{w})$ is a primary of dimension (weight) $(\bar{h},h)$ if

$$\phi_w(w,\bar{w}) = \left(f'(w)\right)^h \left(\bar{f}'(\bar{w})\right)^{\bar{h}} \phi_{\cancel{f(w)}}\left(f(w),\overline{f(w)}\right)$$

$\phi_{f(w)}$: ϕ is measured in $f(w)$ coordinate system.

$\phi(w) \equiv \phi_w(w)$, $\phi(z) \equiv \phi_z(z)$

Infinitesimal conformal trs: $w \to w + \epsilon(w)$
$\bar{w} \to \bar{w} + \bar{\epsilon}(\bar{w})$

is generated by:

$$\oint \epsilon(w)\, T(w)\, dw + \oint \bar{\epsilon}(\bar{w})\, \bar{T}(\bar{w})\, d\bar{w}$$

$\epsilon(w)\, T(w)\, \phi(w_0, \bar{w}_0) \to$ singularities

in this product knows about

$$\left[\oint \epsilon(w)\, T(w)\, dw, \phi(w_0, \bar{w}_0) \right]$$

$\hookrightarrow$ conformal trs. of ϕ

Ex. $T(w)\,\phi(w',\bar{w}') \simeq \frac{h}{(w-w')^2}\,\phi(w',\bar{w}') + \frac{1}{w-w'}\,\partial_{w'}\phi(w',\bar{w}')$

$\bar{T}(\bar{w})\,\phi(w',\bar{w}') \simeq \frac{\bar{h}}{(\bar{w}-\bar{w}')^2}\,\phi(w',\bar{w}') + \frac{1}{\bar{w}-\bar{w}'}\,\partial_{\bar{w}'}\phi(w',\bar{w}')$

Ex. $T(w)\,b(w') \simeq \frac{2}{(w-w')^2}\,b(w') + \frac{1}{w-w'}\,\partial_{w'}b(w')$

?

$-2b\partial c + c\partial b$ $\Rightarrow$ b is a primary of weight $(0,2)$.

Ex. c is a primary of weight $(0,-1)$

$\bar{b}$ " " " " " $(2,0)$

$\bar{c}$ " " " " " $(-1,0)$

∂X^μ " " " " " $(0,1)$

$\bar{\partial} X^\mu$ " " " " " $(1,0)$

$(\bar{h},h)$ Using these results we can convert the mode expansions to z coord.

$$z = e^{w}$$

Ex.: $b(w) = \sum_n b_n e^{-nw} \Rightarrow b(z) = \sum_n b_n z^{-n-2}$

$$\bar{b}(\bar{z}) = \sum_n \bar{b}_n \bar{z}^{-n-2}$$

$$c(z) = \sum_n c_n z^{-n+1}$$

$$\bar{c}(\bar{z}) = \sum_n \bar{c}_n \bar{z}^{-n+1}$$

Not all operators are primaries.

Ex. If ϕ is a primary of dim $(\bar{h}, h)$ then $\partial\phi$ is not a primary.

Even for non-primaries we can assign conformal dimensions.

If an operator transforms as a primary of weight $(\bar{h}, h)$ under $w \to \lambda w$ then its weight is $(\bar{h}, h)$.

Ex. If ϕ is a primary of weight $(\bar{h}, h)$ then $\partial\phi$ has weight $(\bar{h}, h+1)$

Ex. $T(\omega_1)\, T(\omega_2) \simeq \dfrac{D-26}{2(\omega_1-\omega_2)^4} + \dfrac{2}{(\omega_1-\omega_2)^2} T(\omega_2) + \dfrac{1}{\omega_1-\omega_2} \partial_{\omega_2} T(\omega_2)$

use expression in terms of b, c, X

D: # of values taken by μ. (space-time dimension)

Similar for $\bar{T}\bar{T}$ product

D: from $T^X = -\eta^{\mu\nu} \partial X^\mu \partial X^\nu$

-26: " b, c part.

$\Rightarrow$ For $D = 26$ T is a primary of weight $(0,2)$

$\bar{T}$ has weight $(2,0)$

D=26 is needed for consistency.
We work with D=26 from now on.
(We'll see later how to connect
this to 4D space-time)
D-26 is known as "central charge"

$$T(w) = \sum L_n e^{-nw} \Rightarrow T(z) = \sum L_n z^{-n-2}$$

$$\bar{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}$$

From now on we work in z
coordinate system but the Hilbert space
interpretation will still be on
the cylinder.

Defn. $|0\rangle$ is a special state such that $\phi(z,\bar{z})|0\rangle$ is finite for any finite z and any local operator ϕ.

e.g. $b(z)|0\rangle = \sum_n b_n z^{-n-2}|0\rangle$

Finiteness at $z=0 \Rightarrow b_n|0\rangle = 0$ for $n \geq -1$

Ex. Similarly $\bar{b}_n|0\rangle = 0$ for $n \geq -1$

$c_n, \bar{c}_n|0\rangle = 0$ for $n \geq 2$

$\alpha^\mu_n, \bar{\alpha}^\mu_n|0\rangle = 0$ for $n \geq 1$, $p^\mu|0\rangle = 0$

$X^\mu : i p^\mu \tau + \cdots$

Ex. $L_n, \bar{L}_n |0\rangle = 0$ for $n \geq -1$.

follows trivially from mode expansion of $T, \bar{T}$ but also by expressing $T, \bar{T}$ in terms of $X^\mu, b, c, \bar{b}, \bar{c}$ and then using the mode expansions of these fields.

State operator correspondence

For every local operator $V(z,\bar{z})$ there is a state $|V\rangle = V(0)|0\rangle$ and vice versa.

e.g. $c(z) \leftrightarrow c_1|0\rangle$, $\partial c(z) \leftrightarrow c_0|0\rangle$

$\partial X^\mu(z) \leftrightarrow \frac{i}{\sqrt{2}}\alpha^\mu_{-1}|0\rangle$ etc.

Ex. $e^{ik\cdot X}$ is a primary of wt. $\left(\frac{k^2}{4}, \frac{k^2}{4}\right)$

Compute OPE with $T(z) = -\eta_{\mu\nu}\partial X^\mu \partial X^\nu$.

$$\partial X^\mu(z)\, X^\nu(w) \simeq -\frac{1}{2(z-w)}\eta^{\mu\nu}$$

Define $|k\rangle = e^{ik\cdot X(0)}|0\rangle = e^{ik\cdot x}|0\rangle$

k: space-time momentum.

In the CFT the complete basis of states is obtained from $b_n, c_n, \alpha^\mu_n, \bar{b}_n, \bar{c}_n, \bar{\alpha}^\mu_n$ acting on $|k\rangle$

$\Leftrightarrow$ complete basis of operators is obtained from products of (derivatives of) $b, c, \partial X^\mu, \bar{b}, \bar{c}, \bar{\partial} X^\mu$ and $e^{ik\cdot X}$.

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Def: For any local operator $V(z,\bar{z})$ we define $f \circ V$ as the conformal transform of V under $z \to f(z)$

e.g. if V is primary of wt $(\bar{h}, h)$ then $f \circ V(z) = (f'(z))^h \left(\overline{f'(z)}\right)^{\bar{h}} V(f(z), \overline{f(z)})$

Def: $I(z) \equiv -\frac{1}{z}$

Define $\langle 0|$ such that $\langle 0| I \circ V(z,\bar{z})$ is finite for all finite z and all local operators V.

e.g. $\langle 0| I\circ b(z) = \langle 0| \sum_n b_n z^{n-2} (-1)^n$

$\Rightarrow \langle 0| b_n = 0$ for $n \leq 1$

Similarly $\langle 0| \bar{b}_n = 0$ for $n \leq 1$

$\langle 0| c_n, \bar{c}_n = 0$ for $n \leq -2$. $\Big\}$ Ex.

Given a state $|V\rangle = V(0)|0\rangle$, we define BPZ conjugate state $\langle V|$ as $\langle V| = \langle 0| I\circ V(0)$.

Ex. Find the BPZ conjugates of $c_1|0\rangle$, $c_0|0\rangle$

Ex. $\langle 0| L_n, \bar{L}_n = 0$ for $n \leq 1$

Defn. of correlation fn.

$$\langle V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n) \rangle$$

$$= \langle 0| R\left(V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n)\right)|0\rangle$$

$\langle 0| L_{\pm 1, 0} = 0$, $\quad L_{\pm 1, 0}|0\rangle$ & similarly for $\bar{L}_{\pm 1, 0}$.

$$\Rightarrow \quad \langle 0| \left[L_{\pm 1, 0}, R\left(V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n)\right)\right]|0\rangle$$

$$= 0$$

$L_{\pm 1}, L_0$. generate infinitesimal conformal trs.

$$z \to z + \epsilon_{-1} + \epsilon_0 z + \epsilon_1 z^2$$

$\Rightarrow \langle V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n) \rangle$ is invariant under this conformal trs.

Ex. Finite version of this trs.

$$z \to \frac{az+b}{cz+d}, \quad a, b, c, d \text{ complex.}$$

$ad - bc = 1$

$SL(2, \mathbb{C})$ trs.

$$z \to \frac{az+b}{cz+d}$$

maps the $\overbrace{\text{complex plane} + \infty}^{\text{Sphere}}$

to complex plane $+\infty$ in a one to one fashion

(Maximal possible fns. with this property)

Reason why this is a symmetry of the correlation fn.