Resurgence of the large-charge expansion

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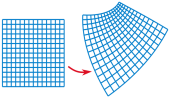
Based on:

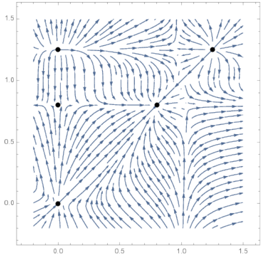
N.A.D, I. Kalogerakis, D.Orlando, S.Reffert [2102.12488]

Introduction

- A conformal field theory (CFT) on $\mathbb{R}^d$ is a quantum field theory with symmetry group

$$\mathcal{G} = \underbrace{\mathcal{Q}}_{\text{internal}} \times \underbrace{SO(d+1,1)}_{\text{spacetime}}$$

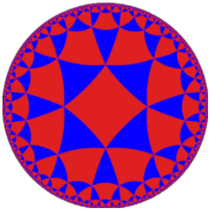




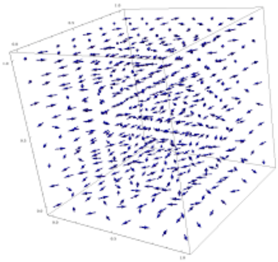
[Picture from Eftychia '18]

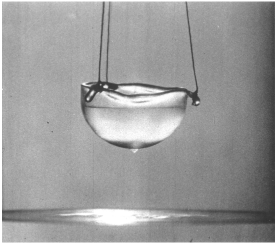
- A CFT can be defined by a list $\{\mathcal{O}_i, \{\Delta_i, \lambda_{ij}^k\}\}$.

- Solution strategies:
  - "Perturbative" methods (Large-N, ϵ-expansion...)
  - Consistency-based methods (Bootstrap, sum rules...)



The O(2) CFT in three dimensions





- Consider the $O(2)$-vector CFT on $\mathbb{R}^3$ (3d XY model, ^{4}He superfluid...).
- Strongly coupled IR Wilson-Fischer fixed point of the $O(2)$ scalar ϕ^4 theory.
- Scaling dimensions Δ_i are contained in the partition function on $S^1_\beta \times S^2$:

$$\mathcal{Z}_{S^2}(\beta) = \mathrm{Tr}\left\{e^{-\beta \hat{H}_{S^2}}\right\} = \sum_i e^{-\beta\Delta_i}$$

$$\mathcal{L}_{\mathrm{UV}} = (\partial\phi)^2$$

$$\mathcal{L}_{\mathrm{int}} = g(\Lambda)\,\phi^4$$

$$\mathcal{L}_{\mathrm{IR}} = \dots$$

Large-charge expansion

Consider the partition function at fixed charge Q:

$$\mathcal{Z}_{S^2}(\beta, Q) = \mathrm{Tr}\left\{e^{-\beta \hat{H}_{S^2}}\delta(\hat{Q} - Q)\right\} = \sum_{Q_i = Q} e^{-\beta \Delta_i(Q)}$$

In the limit $Q \gg 1$ the partition function $\mathcal{Z}_{S^d}(\beta, Q)$ can be realised via an EFT of Goldstone bosons (GB) realising the symmetry breaking:

$$SO(d+1,1) \times O(2) \longrightarrow SO(d) \times D'$$

with natural cutoff $\Lambda \sim Q^{1/d}/r_{S^d}$.

This pattern can be realised in different ways:

- "Conformal superfluid" $\Rightarrow$ Simplest option
- Fermi liquid
- More exotic possibilities...

[Hellerman, Orlando, Reffert, Watanabe '15] [Monin, Pirtskhalava, Rattazzi, Seibold '16]

The superfluid prediction

- D, Q broken, but the combination $D' = D + \mu Q$ is preserved.
- Low-energy modes for this pattern are described by the EFT:

$$\mathcal{L} = c_{3/2}(\partial_\mu \chi \partial^\mu \chi)^{\frac{3}{2}} + c_{1/2}(\partial_\mu \chi \partial^\mu \chi)^{\frac{1}{2}} R + \ldots$$

- The spectrum contains a non-relativistic GB:

$$\chi(\tau, x) = \mu\tau + \pi(\tau, x)$$

$\mu\tau$: vacuum configuration

$\pi(\tau, x)$: GB with $\omega_k = |k|/\sqrt{2}$

- In this realisation one finds the scaling dimension of the lowest Q-primary:

$$\Delta(Q) = \hat{c}_{3/2} Q^{\frac{3}{2}} + \hat{c}_{1/2} Q^{\frac{1}{2}} + \mathcal{O}(Q^{-\frac{1}{2}})$$

[Delacretaz, Endlich, Monin, Penco, Riva '14] [Gaumé, Orlando, Reffert '20 (review)]

Motivation for present work

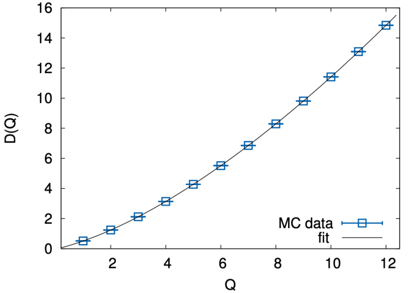
$$\Delta(Q) = \hat{c}_{3/2} Q^{\frac{3}{2}} + \hat{c}_{1/2} Q^{\frac{1}{2}} + \# \begin{Bmatrix} Q^0 \\ \log Q \end{Bmatrix} + \mathcal{O}(Q^{-\frac{1}{2}})$$

- Predictions in the non-perturbative sector (e^{-Q^α} vs. Q^0, $\log Q$...).

  [Hellerman *et at.* '15, Cuomo '20]

- Extrapolation to small charge operators $Q \sim \mathcal{O}(1)$.

- Explaining the effectiveness of $\Delta(Q) \sim Q^{\frac{3}{2}}$ at low Q in MC data.



[Banerjee, Chandrasekharan, Orlando '17]

Extension to O(2N)

- Consider the extension to $O(2N)$ at leading order in $N \gg 1$.
- Fix the charges $Q_{i=1\dots N}$ of $O(2)^N \subset O(2N)$ and consider the limit

$$Q \coloneqq \sum_{i=1}^{N} Q_i, \quad Q/N \gg 1$$

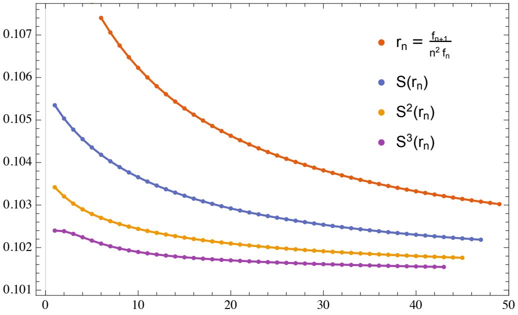
- At leading order in large-N one finds:

$$\frac{\Delta(Q)}{2N} = \underbrace{\frac{2}{3}}_{\hat{c}_{3/2}/\sqrt{2N}} \left(\frac{Q}{2N}\right)^{3/2} + \underbrace{\frac{1}{6}}_{\sqrt{2N}\hat{c}_{1/2}} \left(\frac{Q}{2N}\right)^{1/2} + \sum_{n=0} \hat{c}_n \left(\frac{Q}{2N}\right)^{-n-\frac{1}{2}}$$

Rate of growth?

[Gaumé *et al.* '17, '19]

Large-order growth of $\Delta(Q)$



$$r_n = \frac{\hat{c}_{n+1}}{n^2 \hat{c}_n} \xrightarrow[n\to\infty]{} \text{const.} \quad \Longrightarrow \quad \hat{c}_n \sim (n!)^2$$

One expect instanton-type contributions of the order $\sim e^{-\#\sqrt{Q}}$.

Resurgence and transseries

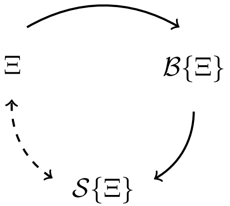
Employ a *transseries* ansatz: $\Delta(Q) \xrightarrow[Q\to\infty]{} \Xi(\sigma_k, Q)$

$$\Xi(\sigma_k, Q) = \Phi^{(0)}(Q) + \sum_k \sigma_k \, e^{-(A_k Q)^{1/\beta_k}} \left\{ Q^{\frac{b_k}{\beta_k}} \, \Phi^{(k)}(Q) \right\}$$

- $\Phi^{(0)}(Q)$: Perturbative series
- σ_k: Transseries constant $\in \mathbb{C}$
- $e^{-(A_k Q)^{1/\beta_k}}$: kth-"instanton" contributions
- $\left\{ Q^{\frac{b_k}{\beta_k}} \, \Phi^{(k)}(Q) \right\}$: kth-"instanton" fluctuation series

$$\Phi^{(k)}(Q) = \sum_n c_n^{(k)} Q^{-n},$$

$$c_n^{(k)} \sim (\beta_{k+1} n)!$$



[Ecalle'81 ... Dorigoni '14 (review)]

Grand canonical picture

Canonical: (Q, V, T) Grand canonical: (μ, V, T)

$$\mathcal{Z}^{(c)}_{S^2} = e^{-\beta \mathcal{F}} \quad (\mathcal{F} = \Delta / r_{S^2}) \qquad\qquad \mathcal{Z}^{(gc)}_{S^2} = e^{-\beta \omega}$$

Legendre transform $\mu \leftrightarrow Q$

$$\omega(\mu) = \sum_{\ell=0}^{\infty} \Bigg[\underbrace{\ell(\ell+1) - \mu^2 - \frac{1}{4}}_{\text{dispersion relation on } S^2} \Bigg]^{-\frac{1}{2}} \longrightarrow$$

ζ – function regularisation

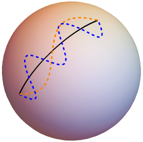
Schwinger trick

...

Heat kernel trace

Heat kernel trace on S^2

$$\langle y | e^{-t\Delta_{S^2}} | x \rangle \simeq$$

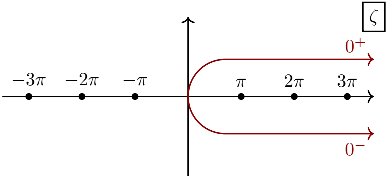


(I) Has a small-t asymptotic expansion:

$$\mathrm{Tr}\left\{ e^{\left(\Delta_{S^2} - \frac{1}{4}\right)t} \right\} \xrightarrow[t\to 0]{} \frac{1}{t} \sum_{n=0}^{\infty} (-1)^{n+1} (1 - 2^{1-2n}) \frac{B_{2n}}{n!} t^n$$

(II) The Borel transform is particularly simple:

$$\mathcal{B}(\zeta) = \frac{1}{\sqrt{\pi}} \frac{\zeta}{\sin \zeta}$$



Prediction for non-perturbative contributions

- The Heat Kernel has a a Borel summation of the form

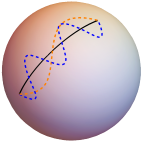
$$\mathcal{S}_{\pm}(\sigma_k^{\pm}, t) = \frac{2}{\sqrt{\pi} t^{\frac{3}{2}}} \int_0^{e^{i0^{\pm}}\infty} \mathrm{d}\zeta \, e^{-\zeta^2/t} \left(\frac{\zeta}{\sin\zeta}\right) + 2i \left(\frac{\pi}{t}\right)^{\frac{3}{2}} \sum_{k\neq 0} \sigma_k^{\pm} (-1)^{k+1} |k| \, e^{-\frac{\pi^2 k^2}{t}}$$

- The leading non-perturbative correction are of the form:

$$\Delta(Q) \supset e^{-2\pi|k|\sqrt{\frac{Q}{2N}}}, \quad k \in \mathbb{Z}$$

- These contributions are of order $\sim 10^{-2}$ at $Q = 1$. Compatible with MC result (first 3 terms fit with relative error within 10^{-2}).
- What about the constants σ_k?

Heat Kernel as a Worldline path integral

$$\langle y|e^{-t\Delta_{S^2}}|x\rangle \simeq$$  $$= \int_{x(0)=x}^{y(t)=y} \mathcal{D}x^\mu \, e^{-\frac{1}{4}\int_0^t \mathrm{d}\tau \, g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}$$

[Strassler '92, Schubert '96 (review) ... Bastianelli '05]

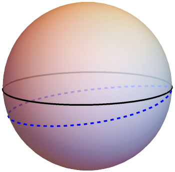
Expectation: saddle-point approximation around closed S^2 geodesics.

... Are these stable saddles?

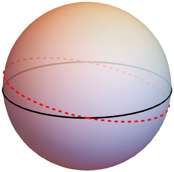
Fluctuation modes on S^2

$$\mathrm{Tr}\left\{e^{t\Delta_{S^2}}\right\} \xrightarrow[t\to 0]{} \frac{1}{t}\left\{1+\ldots\right\} \pm i \left(\frac{\pi}{t}\right)^{\frac{3}{2}} \sum_{k\neq 0}(-1)^{k+1}|k|\, e^{-\frac{\pi^2 k^2}{t}} \left\{1+\ldots\right\}$$

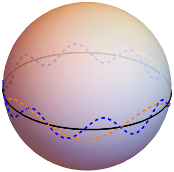
Negative modes
(Borel ambiguities)



Zero mode
(Anomalous scaling)



Positive modes
(Instanton fluctuations)



Summation of the grand potential

- Comparing with the resurgence answer one can fix all ambiguities as

$$\sigma_k^{\pm} = \mp \frac{1}{2} \quad \forall k \in \mathbb{Z}$$

- The grand potential can be extrapolated to any (small) μ by

$$\mathcal{S}\{\omega\}(\mu) = \frac{1}{\pi}\left(\mu^2 - \frac{1}{4}\right) \text{P.V.} \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta \sin\zeta} K_2\left(2\zeta\sqrt{\mu^2 - \frac{1}{4}}\right)$$

- This can be compared with the small-μ expansion:

$$\left|\omega_{\text{small}-\mu} - \omega_{\text{resurgence}}\right|(\mu = 0.65) \;\sim\; 10^{-11}$$

A glimpse of finite-N

- Under the assumptions:
  - $\Delta(Q)$ has an asymptotic perturbative expansion for any N.
  - The leading singularity is determined via saddle of a WL integral for a particle with mass $m \sim \mu$ (Gapped goldstone? [Nicolis, Piazza '13])
- Together with scale invariance, this would lead to a prediction

$$\Delta(Q) \supset e^{-c(2\pi r_{S^2})\times\Lambda}, \quad \Lambda \sim \sqrt{Q}/r_{S^2}$$

- Generalise $\Delta(Q)$ to a transseries of the form

$$\Delta(Q) = Q^{\frac{3}{2}} \sum_n \hat{c}_n Q^{-n} + e^{-c(2\pi)\sqrt{Q}} \left\{ Q^{\kappa} \sum_n \hat{c}_n^{(1)} Q^{-n/2} \right\} + \ldots$$

...Work in progress!

Conclusions

In the the $O(2N)$ Wilson-Fischer CFT at fixed charge(s) Q:

- The regime $Q/(2N) \gg 1$ is asymptotic, with factorial growth $\sim (n!)^2$.
- The factorial growth is driven by WL instantons which are (unstable) saddles of a QM path integral.
- The set of non-perturbative corrections depends only on the S^2 geometry, in particular the leading contributions are of the form $\Delta \supset e^{-2\pi|k|\sqrt{Q/(2N)}}$.

Thank you for your attention!